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THE
ELEMENTS
OF
EUCLID,
WITH
DISSERTATIONS,

INTENDED

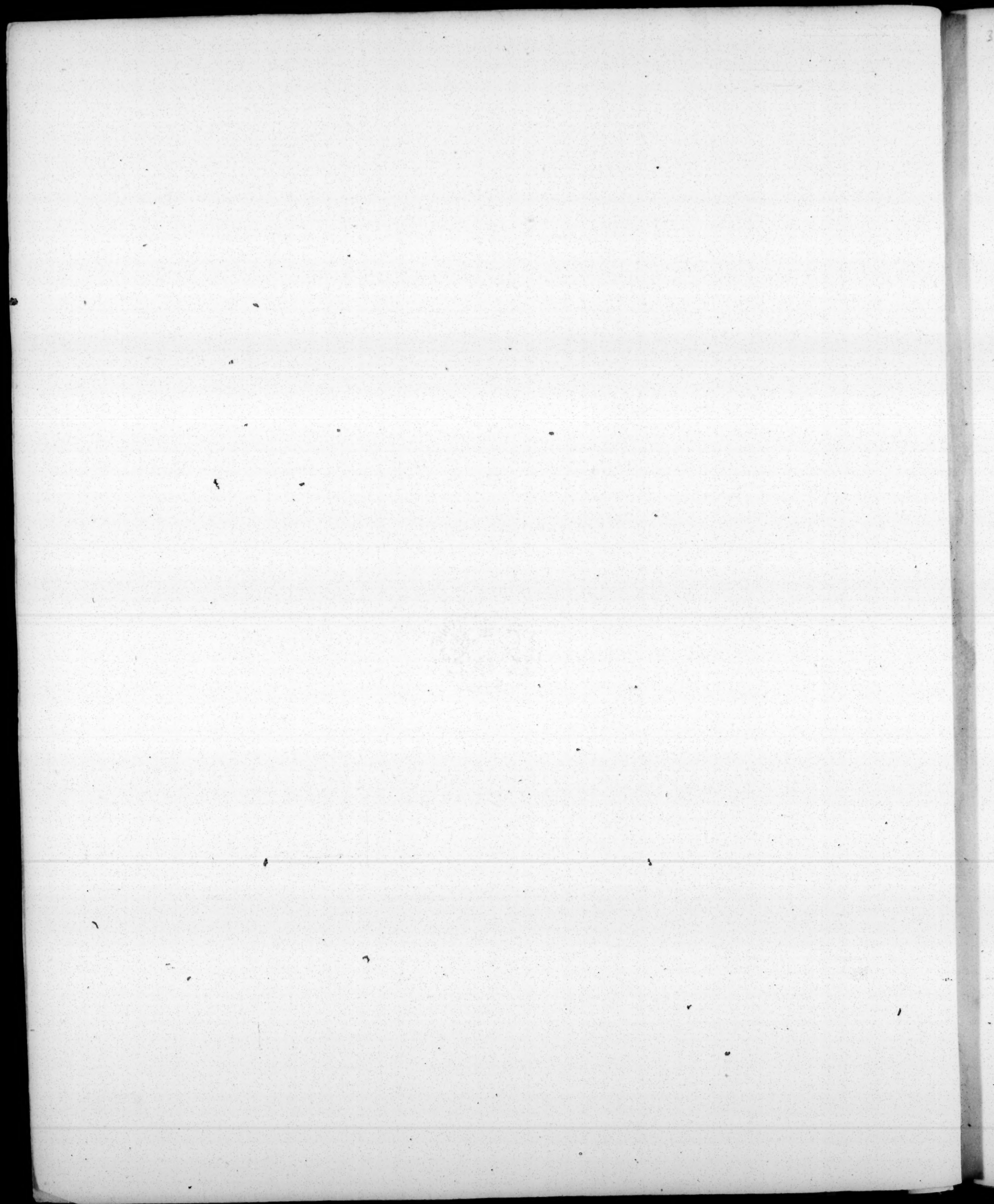
To assist and encourage a critical examination of these
Elements, as the most effectual means of establishing
a juster taste upon mathematical subjects, than that
which at present prevails.

VOL. I.

By JAMES WILLIAMSON, M.A.
FELLOW OF HERTFORD COLLEGE.

*Sed nil dulcius est, bene quam munita tenere
Edita doctrina Sapientum templa serena.* LUCR.

OXFORD:
PRINTED AT THE CLARENDON PRESS.
MDCCXXXI.



TO

RICHARD BURKE, Esq.

DEAR SIR,

IT would be a very entertaining speculation, and well calculated to shew the importance of geometry, to enquire into the probable state of human affairs at present, upon a supposition that the mathematical sciences had been generally known and cultivated in the early ages of the world.

If they had only understood how to make a *map* or an *almanack*, a great deal of rational curiosity would have been gratified, which we now endeavour in vain to satisfy. For mankind have slept on, from age to age, without taking notice of time or occurrences; and as to what concerns *geography* they have been equally remiss. The human race have migrated from one country to another, until they have covered the face of the whole Earth, without gaining the least information by their travels; having rather strayed like cattle, than changed their habitation like rational creatures.

However the happy effects of a proper cultivation of this science, are not confined to *maps* and *almanacks*: for it is by the judicious application of mathematical knowledge, that human nature, notwithstanding its frailty, is rendered superiour to every obstacle, for our efforts keep pace with all our rational imaginations: and thus

it

it rises greatly above all its imperfections, acquiring a dignity which must astonish every one, when he compares what has been done, with the weakness and imperfection of any single individual. And what this might have produced through a length of ages, operating in every quarter of the globe is not so easily to be conceived.

It has indeed been alledged, that the improvements ascribed to the cultivation of this science, have been often the effect of chance, and not produced by any rational scheme of improvement, conducted upon scientific principles. But the contrary appears from undoubted matter of fact; because we every where find that discoveries are made among the most enlightened nations, and never among barbarians. Events, no doubt, will happen, and natural appearances will keep their regular time; but the wild beasts of the field are as likely to make a proper use of them, as illiterate savages. And the little attention paid by ignorant men to those very circumstances which inform and enlarge the understanding, shew us better than any thing else, what human nature can rise and fall to.

Both you and I are sufficiently sensible of the great utility of mathematics in all the affairs of life; and that those whom it concerns are, to a certain degree at least, sufficiently attentive to it. And we only used to regret that the world is ignorant, how successfully this science might be employed for making us rational creatures.

It has often been the subject of our conversation and surprize to run over the various methods contrived for making mathematicians, without imposing upon them the necessity of being rational creatures. And such discourses have generally ended in your urging me to the execution of this plan; the first part of which I here beg
leave

leave to present to you. I have had sufficient marks of your friendship to be sensible that you will be much more solicitous about its success than I am myself: and from your partiality to me, I am also persuaded you will think that I have not done it justice in the execution.

It is true I could have improved the style very much; but it seems to answer my purpose better in its present form: for I write not to make people read, but to make them think. I have also affected a familiarity of phrase, to engage the attention of the reader by expressing geometrical ideas in common language.

But whatever your opinion of the work may be, I beg of you to accept of this address, as a testimony of the high esteem which I have for your character and abilities. It is with the greatest pleasure that I recollect the share which I have had in your education, which was carried on, not in the person of master and scholar; but rather in the character of two friends who had taken a somewhat different view of the same subject.

I am, DEAR SIR,

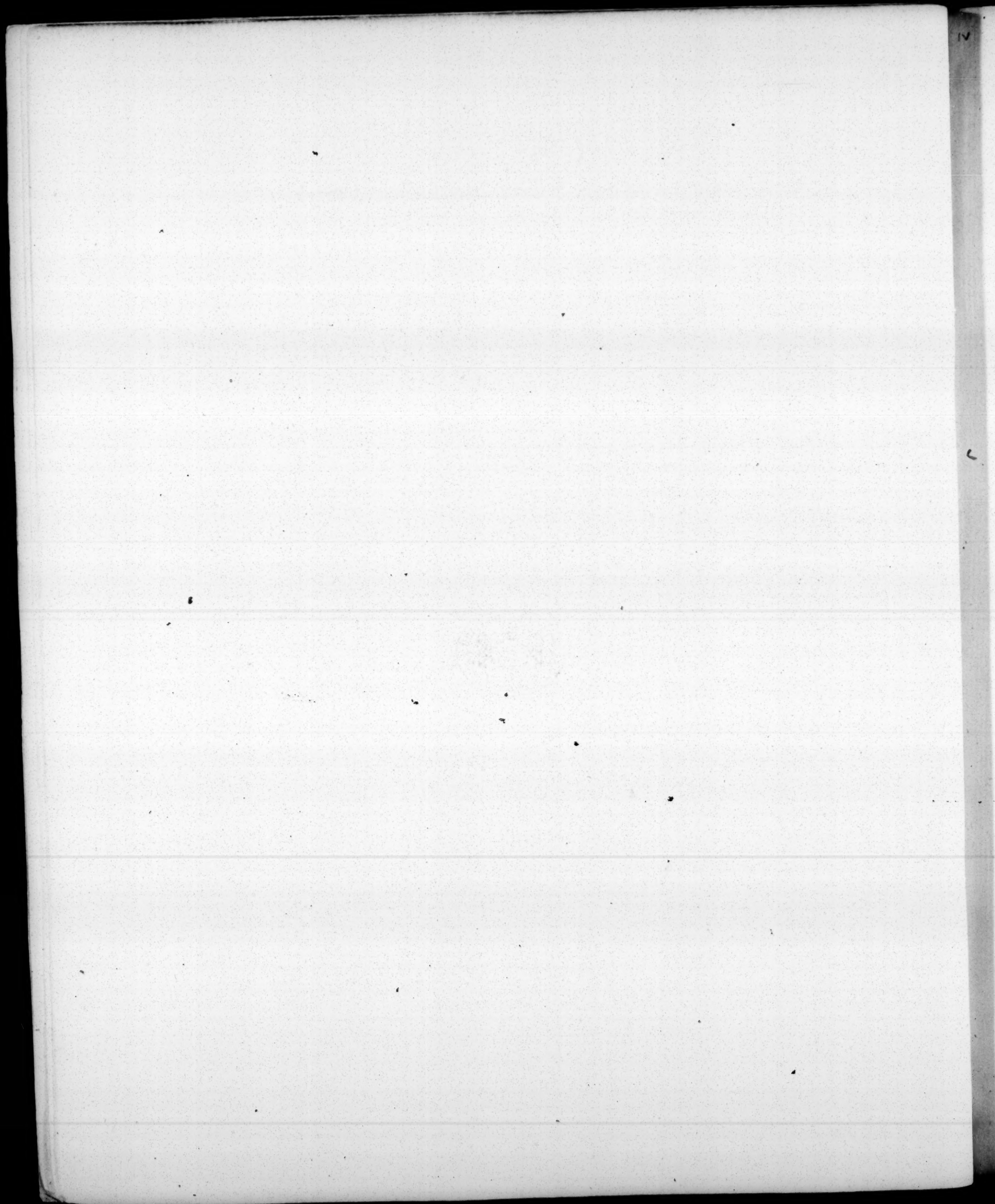
with the greatest respect,

Your most obedient

humble servant,

HERTFORD COLLEGE:
March 27. 1781.

JAMES WILLIAMSON.



DISSERTATION I.

IT is my intention in this dissertation, after saying something concerning the use of commentaries, to conduct the learner, step by step, to a solid, rational and extensive view of the principles of geometry; by explaining their original and improvement through several different gradations, until they put on a scientific form; reducing what I have to say under distinct heads, for the benefit of the ignorant and thoughtless reader.

CHAP. I.

Of the necessity and use of commentaries.

THE dulness of commentators is a subject of such general complaint, that it may be proper to inquire how it comes to pass that books are not written in such a manner as to make a commentary useless or impertinent: and this inquiry may be proposed with the greater propriety, as the labours of commentators, besides their dulness, seldom or never answer the expectations of the public in other respects; which they no doubt are apt to imagine the author himself could have fully gratified, by condescending a little to the weakness of his readers, and rescued his works out of such clumsy hands.

What apology therefore shall I be able to make for loading with a commentary the most perfect book in the world! But though it may not be agreeable to that general delicacy which an author is bound to preserve towards his readers; yet it may nevertheless be proper to inform them, that they seem to be rather partial to themselves upon the present question; never considering how much of

the blame ought in all conscience to be laid at their own door : for though it may be a secret to a great many, yet it is an undoubted truth that the most perfect writer requires that his work should be put into the hands of those who can read. And my intention in this edition is not to correct my author, but to supply a defect which it could not have been very consistent with his plan to remedy. For he has written his book expressly upon the supposition that his reader was endued with the faculty of attention : and as this is a disposition of mind with which the book is but rarely taken up, though it will always be laid down with it, or else it has been used to little purpose : a few seasonable warnings therefore, to rouse the attention of the indolent reader, may be given with great propriety, and without bringing any reflexion upon the character of the author, who in point of perspicuity, excels every one.

But I will even go farther and venture to affirm that an author, who writes upon subjects of science, may find it often by no means convenient to deliver himself in such a manner as to be always intelligible even to those whom he would wish to have for readers. Because authors are confined to a particular method of arrangement, if it be their intention to deliver opinions or discoveries in a systematic manner. And although there has been a senseless and incessant clamour, for almost two centuries, against this method ; yet it seems to be the only way of keeping knowledge within such limits, as to be by any means manageable by the human mind.

It is true indeed that our progress in acquiring knowledge, when left to ourselves, and our own experience, is directly contrary to this, being first condemned to an examination of particulars. And even when we take up with the systems of others, it is requisite that we have laid in a sufficient stock of materials to enable us readily to comprehend any very general doctrine. For no general rule, or law, or theorem, or what you please to call it, is by any means to be understood, unless we have particular instances ready to apply, as occasion requires : always however excepting a *genius* of the kind given to *Hudibras* in the following lines, containing the meaning and importance of many volumes.

His notions fitted things so well,
That which was which he could not tell.

The

The fault therefore is not in the systems, but arises from the general incapacity which mankind seem to labour under, for judging of the merits of such as have been offered to their consideration; and as there are quacks of all denominations, ever lying in wait to take advantage of the simplicity of the multitude; the world by this means has been over-run with counterfeits.

To have a genius for any science seems to me to imply a readiness at finding out particular instances to apply to any general rule. It is therefore to be suspected that those who are deficient in this kind of invention will by no means find their progress in any science, answerable to the time which they bestow upon it: they may commit the rules or the theorems to memory and nevertheless be ignorant of their meaning and application. Hence a certain degree of invention becomes necessary even for the ready acquisition of a science; and this perhaps not so different from that kind of invention by which the principles were at first discovered, as many have been apt to imagine.

Now here is the difficulty; a scientific book ought to contain knowledge in that compact form; in which every one would chuse to take a review of it, after he has made himself master of the subject: and not incumbered with all those particular instances, which would now stand in the way of the readers imagination as much as they assisted it before. For instance, who could relish the noblest of Newton's theorems if it presented itself to his mind encumbered with all those properties, which had led him from common notions to the right understanding of so sublime a speculation.

A book therefore, if this reasoning be just, which would be proper for a learner would be fit for nobody else: and the greatest perfection of writing will still leave occasion and employment for the talents of that kind of commentator, which I profess myself to be. The sweepings of the author's study would furnish the best and most authentic materials for works of this kind; and yet humble as the office may seem, if I can execute my task but nearly up to the idea which I have of it, I shall not regret my labour. The reader however will be disappointed, if he expect to find Euclid either corrected or enlarged, my purpose being nothing more than

to conduct the learner to that sense in which the author wished himself to be understood. An ostentation of learning is a fault which prevails among many commentators; for they seem to be apprehensive least the world should think they know nothing but their business: and therefore instead of explaining their author, endeavour to persuade the world that they understand the subject better than he himself did. But I do not write to the strength but to the weakness of mankind; and therefore would not chuse to be considered as challenging the whole world to find fault, but only as applying a remedy to a weakness which my own experience has found does exist.

CHAP. II.

Concerning the Original of the geometrical principles.

QUANTITY of both kinds, extension and number, is always forcing itself upon us whether we will or not; and must therefore leave some very fixt and determinate impressions upon the minds of the most inconsiderate. We cannot stretch out our hand without receiving a perception of extension; nor open our eyes without seeing figured objects, bringing along with them to the mind a consciousness that they are more in *number* than one. This begets notions, which are by no means peculiar to any single art or profession, but common to all men. The geometrician selects the most accurate of these, and with such materials lays the foundation of his science. Particular circumstances have rendered some notions concerning quantity more invariable than others; or rather so fixt that nothing can alter them. As it would discover ridiculous affectation and ignorance to pretend to change these, which is indeed impossible, so it is also below the dignity of our author to affect to disguise them by any forced or unnatural construction, to make them wear a more philosophic appearance. But although the twelve common notions which he has selected, are to be understood according to the vulgar conception of them; yet the learner must give them a very particular examination: because it is not sufficient to have these notions, but we must also have the ready use of them

them. And this may be acquired in some such manner as the following.

Let the student provide himself with a ruler and compasses, and after some practice in drawing straight lines, and describing circles; he is next to proceed to the examination of the common notions, as if they were properties of straight lines only, and true of nothing else. For without this precaution he will undoubtedly be liable to have the distich quoted in the last chapter applied to him. And any tincture of the *budibrastic genius* disqualifies a man for this science; and excludes him from a great deal of rational amusement, to say nothing of more solid advantages. I shall therefore at the porch, not only lend the learner my advice but also my assistance in stripping himself of those prejudices which would disgrace his behaviour after he has been admitted into this magnificent temple where all the wonders of the world are displayed.

The reader may believe that I never would have introduced this advice with so much form and circumstance, without a firm persuasion that it is of the last importance. He is therefore immediately to set about the work, by describing a circle, not a geometrical but a mechanical circle; and such as any ordinary compasses will exhibit; drawing at the same time several straight lines from the center to the circumference. He is next to satisfy himself of the equality of these straight lines, by measuring them with his compasses: his conclusion will be, that they are equal; and he will find his opinion of their equality grounded upon the first common notion; because they are all equal to the same length, viz. the distance between the extreme points of his compasses. But it is carefully to be observed that this is not to be made the subject of a transient reflexion, but of frequent and close meditation; varying the center and radius to the utmost limits of the compasses; with now and then a thought upon the limited nature and imperfection of the instruments.

The second and third of the common notions may be examined by describing two circles with the same center, but at different distances, and drawing straight lines from the center to the remotest circumference; the parts of the straight lines intercepted between the two circumferences are equal; and will illustrate the
second

second common notion by taking the less radius from the greater. And thus we are to proceed untill we have satisfied ourselves that these common notions are true at least of such straight lines as we can draw upon a piece of paper.

I beg the reader's pardon for my impertinence; but he is farther to be admonished, that it is not sufficient to run these things over in his own mind; but that he must be able to express them to the conviction of a by stander; and this will make it necessary to distinguish his lines and circles by the letters of the alphabet.

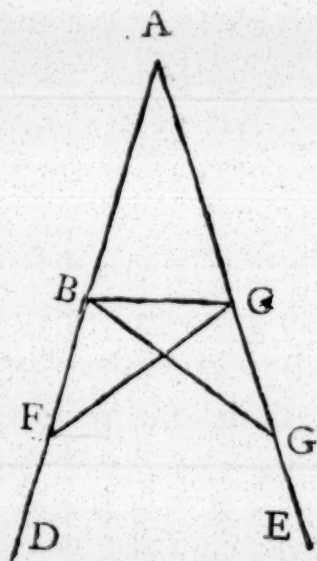
CHAP. III.

The same subject continued.

SUPPOSING this business of the straight lines accurately discussed; the learner is next to shut his compasses; and then observe their progress in opening until they take the direction of a straight line: during this operation, he will find the inclination of the legs continually varying: at first nothing, then gradually increasing until it disappears when the legs become one straight line. This inclination is a quantity, though not a tangible substance, but this the reader will do well to convince himself of; and for this purpose he may observe that any particular inclination may be equal to another, or the half or the third part of it. But the common notion of this kind of quantity is not so regular or determinate as that of a straight line; though it exhibits every possible shape which it can take in opening the compasses as above directed: the reader therefore will be pleased to instruct himself properly in this: and then proceed to examine whether the common notions are not also true when applied to this kind of quantity.

And for this purpose I would recommend a triangular piece of wood, of the shape of a right angled triangle with unequal sides, being afraid to meddle with circular arches, least we should *conjure up* a prejudice which we might want art afterwards to *lay*. By the assistance of this triangular piece of wood, make two equal inclinations (or angles) upon paper, taking care to make the lines unequal, to prevent prejudice. After these are made, their equality may

may be inferred from the first common notion, as each of them will be equal to the inclination of the two sides of the peice of wood : add to these two equal angles, other two equal angles ; which may be done by the assistance of a different corner of the same piece of wood ; and this will illustrate the second and third ; according as you consider one of them as taken away from the whole angle made up of the two ; or as added together to make one. But it will be necessary previous to this, to acquire a ready and accurate way of expressing the different inclinations of lines, (called angles) by the letters of the alphabet. The figure annexed will be a very proper one for practice and the task which I would set the reader is to tell the number of angles and the different methods of expressing them ; giving him to understand that their number is above fourteen ; and that, CAB, CAF, CAD ; GAB, GAF, GAD ; EAB, EAF, EAD ; BAC, BAG, BAE ; FAC, FAG, FAE ; DAC, DAG, DAE ; are only so many different ways of expressing the same angle ; nor does this great variety, in the least puzzle or perplex the conceptions of an *adept*. This looks so much like a riddle that I think it cannot fail to engage the attention of the curious. But not to trust entirely to the reader's own ingenuity for unraveling this knotty point ; let him observe the following hints ; the letter at the meeting of the lines, whose inclination to one another we want to express, is put in the middle, and it is sufficient that the other two letters, each express some point in each line : thus the inclination of FB to BC is called the angle FBC or CBF : and the inclination of DB to BC is the very same with the other, as is obvious, and is called the angle DBC or CBD : the inclination of BC to CE is called the angle BCE or ECB ; and the inclination of GC to CB is the same with the other and is called the angle GCB or BCG. But farther the angle ABG is made up of two angles viz. ABC, CBG : and the angle ACF is made up of two angles viz. ACB, BCF. And to assist the reader in applying the second common notion I have made the angle ABG equal



equal to the angle ACF : and I have likewise made the angle CBG equal to the angle BCF ; and the conclusion will be that the angle ABC is equal to ACB.

CHAP. IV.

In what manner our common notions begin to take a scientific form.

AFTER the reader has prepared himself according to the directions given in the last two chapters ; it will now be proper to take a review of the instruments, which he made use of, for regulating his conceptions : and these, he will find, were very limited, being confined to a few inches. Let him next ask himself, whether he has any reason to suspect, that the conclusions, obtained by the help of these instruments, were equally limited. Nor is this point to be determined rashly, but so as to be still certain that he treads on firm ground : and we may venture to draw this conclusion for him ; that, without any great force upon his imagination, he can conceive his instruments to have a double or triple extent without finding the least reason to change his opinions. And by proceeding thus, he will certainly come to this conclusion at last ; that although these instruments might be the occasion of his turning his thoughts to this subject, yet his opinions were nevertheless derived from the nature of extension in general ; and that they knew no other limits, but such as bounded extension itself : but more particularly, that a circle whose radius is a thousand miles, or the thousand part of an inch, would furnish the same conclusions as one of two or three inches. Here now our opinions, which before were measured by our instruments, begin to put on a different form and display to our imagination the first dawn of science.

If any one should pretend that he had the notions originally in this very general form to which I have been endeavouring to lead him ; I have only to say, unless they were acquired by an examination of particulars, he will find his notions fit every thing so well, that when he comes to apply them to particular instances, he will not be able to tell *which is which*.

The reader is to endeavour next to get something like a scientific notion of an angle, by correcting the vulgar notion of an angle,
by

by which is understood the corner of any thing. Now this does not so much depend upon any stretch of the imagination, by which large objects, and such as exceed the experience of our senses, are to be made the subject of Contemplation ; because the point where the lines meet, together with any point in each of the lines fixes the angle invariably : or in other words, the three points denoted by the three letters of the alphabet, expressing the angle, fixes any rectilineal angle : for the angle is not changed by making the lines longer or shorter ; but only by opening or shutting them ; conceiving them to turn upon a pin like the two legs of a pair of compasses.

But our instruments are not only too limited for our conceptions, but are inaccurate in other respects. We have a very clear notion of three dimensions viz. length, breadth and thickness : and surely without nicely separating and distinguishing these, it is impossible to have true and proper conceptions of magnitude. But these different dimensions cannot be represented by our instruments. For when we attempt to draw a line or even to mark a point ; our line and point possess all the three dimensions in as great perfection as a cannon ball or the mast of a ship. The human mind, when once made sensible of its powers, will never suffer its conceptions to be so clogged with matter : which has put those who carry their views beyond the vulgar, upon inventing some method by which our conceptions may be rendered more rational and consistent ; and this is the original of definitions.

CHAP. V.

Of definitions.

OUR author has proceeded with singular judgement in laying down his principles : where the common notions are sufficiently distinct and accurate, he has inviolably adhered to them. But when these are too incorrect or too indeterminate, he explains the sense in which he would have any particular term be understood ; and what conception he requires his reader to have of the figures which he defines. Definitions may be considered as of two kinds ; first, such as serve only to explain the meaning of a word ; but these

are not properly geometrical definitions, for from this no consequence can follow unless there be a mystical virtue in the name. Secondly, those from which consequences or properties do follow, which may be called geometrical to distinguish them from the others. The definition of a circle, given by Euclid, is of this kind for all the properties of a circle follow as consequences from it. And here the reader is to be admonished, that, upon this and such like occasions, his common notion of a circle is to be laid aside, and nothing admitted as a property of this figure, unless it can be shewn to follow from this definition.

Though Euclid in the arrangement of his principles has placed the common notions after the definitions, yet they are prior to them in the order of conception; and indeed if this is not attended to, some of his definitions will be unintelligible; for instance his definition of a straight line. He says a straight line is that which lies evenly to the points in itself. Now if I am to conceive nothing previous to this, respecting a straight line; what can I understand by this definition; or what can I infer from it? The reader will be just as much at a loss to conceive the meaning of the word *evenly*, as of the straight line itself. But if we consider this definition as an improvement upon the common notion of a straight line, (see com. not. 12.) every thing is very intelligible: for after a proper examination of this principle, that two straight cannot inclose a space; every body will infer, though not scientifically nevertheless very confidently, that every straight line must lie evenly to all the points in itself; otherwise he certainly might have hopes at least of making two of them inclose a space. I would be rightly understood upon this point; nobody can imagine that it is my opinion, that Euclid intended that the one of these should be inferred from the other scientifically; but only that the definition expresses the conception, derived from two lines, reduced into a more simple form; though indeed he himself reasons from the common notion as will appear in the fourth proposition.

Dr. Hooke one of the greatest improvers of practical mathematics, was notwithstanding so sensible of the necessity of laying an accurate foundation in theory that he thus speaks concerning a point. *A point is that which has no part.* "This which some
" would

“ would deem the most inconsiderable thing in the world, seems
 “ yet the most difficult to be understood ; no sense, or imagination
 “ or fant’cy can reach it, nor words describe it, but by a negative,
 “ to tell you what it is not : For it is not to be taken in the sense,
 “ that the whole Earth is called a point in respect of the Universe,
 “ nor in the sense that the end of a tapering thing is called a
 “ point, as of a pin or needle ; though they seem to be the smallest
 “ things we know ; because these latter may be said to have as
 “ many parts as the fore-mentioned ; for since all quantity is divi-
 “ sible *in infinitum*, the least quantity may be divided as often as
 “ the greatest, and therefore whatsoever is divisible must have
 “ parts, and therefore none of these can be properly called a point,
 “ in the sense here named, unless this point be understood to be
 “ the *apex* of a mathematical cone or pyramid, where the super-
 “ ficies of it is determined, for that will be a mathematical point :
 “ But it cannot be supposed of a physical point, or material cone
 “ or pyramid, for that will have extension and bluntness. And we
 “ find that *microscopes* will make those points divisible even to sense,
 “ nay even almost to discover a new world in a point, nay there is
 “ one now that affirms he has seen more than ten thousand living
 “ creatures in the bigness of a very small sand, which itself indeed
 “ is but a visible point to the naked eye ; and each of those ten
 “ thousand may have worlds within them. We know not the
 “ limits of quantity, matter, and body as to its divisibility or ex-
 “ tension ; no imagination can comprehend the *maximum* or the
 “ *minimum naturæ*, our faculties are finite and limited, and we
 “ must content ourselves within the orb and sphere of their acti-
 “ vity. And acquiesce in a negative definition, and understand if
 “ we can somewhat that is smaller than the smallest, though that
 “ be also improper ; for in that which is not quantity, there is
 “ neither smaller nor bigger, we must endeavour to understand
 “ some what infinitely little, less than which there cannot be,
 “ somewhat that has no bigness or extension or quantity, but only
 “ position and respect to quantities circumjacent : From which, to
 “ this or that body, there is a determinate length and distance ;
 “ and upon this account, wherever we endeavour to understand
 “ this notion, our imagination will represent to us the smallest

“ visible body, as an exceeding fine sand, or a mite, or the point
 “ of a needle, or the smallest visible body we have ever seen on
 “ paper, or the like; which we must be content with, since the
 “ fant’cy forms nothing but what is first in the sense, though it be
 “ none of these. And in truth it can have no true definition that
 “ will reach its essence. Analogous to this *point*, sign or nothing
 “ in quantity is the *nought*, cipher or zero in number: The *never*
 “ in time: The rest or quiet in motion. For as no aggregate of
 “ points will ever produce a line or quantity; so the multiplication
 “ of *noughts* or *ciphers* will never produce a number; and as the
 “ addition of *nevers* cannot make time, so the aggregate of *rests*
 “ cannot produce motion. So that all these may not improperly
 “ be called the *terminus* or bound, from which they all begin; so
 “ quantity may be said to begin from a point or nothing: number
 “ may be said to begin from *nought*, cipher or zero: time may be
 “ said to begin from *never*: and motion to begin from rest. And
 “ as the *minimum naturæ* may be said to be the first quantity; if at
 “ least there be a *minimum* in nature, so an unite may express it in
 “ numbers; *instant* in time, and moment in velocity. It may pos-
 “ sibly be thought I have said too much of *nothing*, but yet it
 “ seems to be of the greatest consideration in nature; for it seems
 “ to be the beginning of every creature; even the greatest creatures
 “ having been traced to begin from an atom or point, no eye or
 “ sense can reach it; nor any understanding limit it; that the be-
 “ ginning of a very large animal hath been seen alive, ten thousand
 “ times smaller than a mite, may be proved, and yet how much
 “ smaller it may have been is not determined.”

It has been said of the musick of the spheres that it is so loud
 we cannot hear it: and of a mathematical point, according to
 this explanation, it may be said, that it is so simple it is impossible
 to comprehend it. This no doubt, must preposses the reader with
 strange apprehensions of difficulty in the prosecution of a science
 which sets out so unaccountably; requiring us to admit as a prin-
 ciple what we cannot comprehend.

I would not have thrown this metaphysical bugbear in the way
 of the reader; only I was afraid lest he might stumble upon it of
 himself, with some hazard to his understanding, if he should pursue
 the

the metaphysical analysis of it, until he had no doubts left but that it was really incomprehensible; especially if he happened at the same time to discover, that the same ingenious subtilties would also apply to lines and surfaces, by which means they might likewise be entitled to a place among the incomprehensibles.

I shall therefore beg leave to call his attention to a method of explanation very different from this, but which arises very naturally from what I mentioned before viz. that the definitions should be considered as subsequent to the common notions; and introduced merely as auxiliaries to them. And in the present instance, we have a very clear notion of three dimensions in magnitude; though we always find them so connected that it is impossible to separate them from one another. Let us therefore try to contemplate them together, but one after another. Every part of space that we consider has a shape; for instance the shape of a room: this space the mathematicians call a solid; that which limits it or gives it this shape is called a surface; which must be considered as having only length and breadth; because if it had thickness also; then it would not be the boundary of the solid, but a part of the solid itself. Therefore the proper definition of a surface will be, that which hath length and breadth. But farther this surface has a shape or is limited; that which limits it cannot have length and breadth; for then it would not be the limit of the surface but a part of it. Therefore the proper definition of a line is length without breadth. Lastly this line has its limits; which limit cannot have length, for then it would not be the limit of the line but a part of it. Therefore the proper definition of a point would be, that which belongs to magnitude, and has no parts, that is none of the common dimensions, length breadth or thickness. Thus it is easy to conceive how these definitions of a point line and surface are derived from the common notions of magnitude.

C H A P. VI.

The same subject continued.

AS the definitions which we have been explaining are never quoted or appealed to as tests by Euclid, the learner might be ready

ready to conclude that they were placed here for no other purpose but to impress one with an idea of the difficulty of this science.

Though it be true that they are never used directly, yet by being placed where they are, they stop the mouths of impertinent critics, who would be ready to start objections which are immediately removed by these definitions.

In this chapter I intend to explain what is meant by a PLANE, a plane rectilineal angle, a right angle and an acute and obtuse angle; first considering what opinion every one might form of these, by examining those instruments or circumstances which would fix his attention to this subject.

To understand rightly Euclid's definition of a plane surface, it will be necessary to have recourse to the former supposition, that the definitions are improvements upon our common notions: for instance, let us suppose one at work, in the manner of a carpenter, to satisfy himself that a surface was even, suppose the surface of a table: he would apply a ruler in all directions from side to side; and finding that it touched the table in every point and in every direction, he would conclude that the surface was even. And now we shall find that the definition is nothing else, but giving the notion acquired by this operation, a regular form and sufficient extent,

For Euclid's plane surface which is commonly called a PLANE is this surface of the table, endued with a scientific evenness instead of a mechanical one; and instead of being confined to any shape; of an unlimited extent. It is *in* or *upon* such a surface as this, that Euclid supposes all his lines and figures to be described and drawn, in his first six books: and this is the more carefully to be observed, because, though it is not mentioned in the demonstrations, yet there are many properties of lines and figures, which are not true, but upon the supposition that the lines are in the same plane, as I shall have occasion to observe.

And the plane rectilineal angle is the inclination of two straight lines to one another *in* or *upon* such an even surface as this, which meet together but are not in the same direction. They might be inclined to one another so that when produced they would intersect
or

or cross one another ; but until they have met they are never considered as containing an angle.

This plane rectilineal angle may be of three different kinds ; according to the different inclinations which two straight lines may have to one another. Take two pens, and run a pin through the extremity of the one, and at the same time, through the other, at some distance from its extremity, so that the one of the pens may make two angles with the other, on each side one. Turn the pens into the same direction, and making one of them revolve, and then observe its progress in revolving until it takes the same direction on the other side : and during this whole revolution there is but one fixed or determinate position of the lines with respect to one another ; and that is when the revolving line has the same inclination to the other on each side ; and these equal inclinations are called right angles ; and the other two have their names according as the angle is greater or less than this.

The reader has been very inattentive to my directions upon this head, if he does not perceive, that there is an unlimited variety both of acute and obtuse angles, but that there is but one right angle.

And hence by the bye it is obvious that Euclid formed his definitions from common notions, as he reckons this one ; that all right angles are equal, probably as a hint to the reader, that he might discover from what source the definitions were derived ; because this common notion may be regularly demonstrated from the definition of a right angle, and therefore must have been placed where it is for some indirect purpose. Upon the whole then we may conclude, that Euclid's definitions are not the imaginary phantoms which the metaphysicians have represented them : but notions derived from material objects, and new modeled, not for the purpose of empty speculations, but, that they might be applied to the same kind of objects which first suggested them, upon a more accurate and enlarged plan.

CHAP. VII.

Of postulates.

THIS science has now acquired such an extent and accuracy, by improving the common notions into definitions ; that our former instruments are now become inadequate to our purposes and views. For our notions both of a straight line and circle are so refined and enlarged ; that no instrument is sufficiently accurate or extensive for performing these two problems, the drawing a straight line and the description of a circle.

Here the case of the science appears desperate ; and the author reduced to the necessity of giving up his high pretensions to accuracy and universality, by suffering the science to fall back again into its original state, as far as respects its instruments at least. And I am certain the artifice used upon this occasion to preserve the dignity of the science is by no means generally understood ; and this will be confirmed beyond contradiction by attending to the foolish and childish reasons generally given for admitting these postulates, alledging that it is so easy to conceive how they may be done, that they are to be admitted without the least scruple or hesitation. That a thing may be easily done is surely a very bad reason for neglecting to give rules for doing it : and if that had been the case in the present instance, I am pretty confident these commentators would have been saved the trouble of their apology. But the opinion of Newton I suppose will be decisive upon this occasion, and we find him expressing himself directly to our purpose as follows. “ *Nam et linearum rectarum et circulorum descriptiones, in quibus geometria fundatur, ad mechanicam pertinent. Has lineas describere geometria non docet sed postulat. Postulat enim ut tyro easdem accurate describere prius didiceret, quam limen attingat geometriæ ; dein, quomodo per has operationes problemata solvantur, docet ; rectas et circulos describere problemata sunt, sed non geometrica. Ex mechanica postulatur horum solutio, in geometria docetur solutorum usus. Ac gloriatur geometria quod, tam paucis principiis aliunde petitis, tam multa præstet.*

After such an authority the judicious reader will be ready to agree with me, that these problems are taken for granted, not
because

cause it is easy to conceive how they may be done, but for a much better reason, because it is impossible to do them by any method consistent with the principles of this science.

But even in this, which must have been a matter of no small difficulty, our author has done every thing in the power of man, to render this unpromising part as correct as possible, by making it put on a very respectable figure indeed, giving it such an air, as discovers but little of its mechanic original,

An ordinary genius would probably have given directions for drawing a straight line and describing a circle, with a ruler and compasses; and when the limited nature of his instruments was mentioned as an objection to his science, and that he could not with propriety extend his conclusions beyond a sheet of paper; he would then probably propose a chain and poles by the assistance of which he might undertake to enlarge the scale of his operations; yet with waggon loads of such instruments he never could hope to produce one scientific construction.

But Euclid has acted with more judgement, and never would be accessory to the opening a door for admitting upon his scientific stage the whole tribe of mechanics, with the several implements of their trade. He throws them two problems to perform, but this is to be done behind the scenes, without exhibiting their mechanical instruments, declaring that upon every future emergency, he is determined to stand or fall by his own principles.

C H A P. VIII.

Of the instruments made use of for communicating geometrical knowledge.

ALTHOUGH language be a general medium for communicating knowledge of every kind; yet particular subjects require the introduction of some auxiliary instruments. Unless the nature of the thing is such, that it cannot fail of itself to make a strong impression upon the mind, or that it is of no great consequence whether it be particularly examined or not, it will be found difficult to command the attention; and particularly, which is absolutely necessary in a chain of reasoning, to make the thoughts of

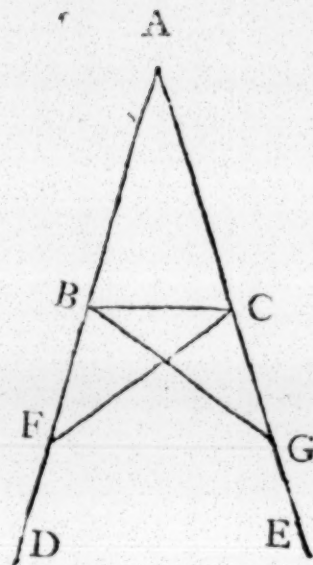
the reader and writer follow each other in the same order, with a certainty that they are both thinking of the same thing at the same time. The art of logic is an instrument of this kind, but the great diversity of opinions prevailing among mankind is a sufficient proof that this instrument is either neglected or misapplied. Indeed that wonderful uniformity of opinion which has been universally found among all nations, and in all the different ages of the world upon geometrical subjects, cannot fail to draw our attention to this singular circumstance; and make the learner desirous to be let into the secret of this harmony, as far as one ignorant of the subject can be supposed capable of entering. And this will be found to proceed from the method of demonstration to which geometricians have tied themselves down; and their being possessed of an instrument, perfectly adequate to the purpose of communicating their knowledge to others, and by which they are able infallibly to examine their own. And this instrument in geometry is figures or diagrams: which Euclid expresses by the different names of *ᾠμα*, *καταγραφή*, *ἔκκ.* The first is his general term, the second he commonly applies to a figure which has been described or at least compleated; and the third commonly to similar figures.

Now these diagrams or figures may be made so exactly to resemble the subject matter of any proposition, that, if we think at all, it is impossible to mistake the order of thinking, which the author has prescribed, or to draw a different conclusion from that, to which he intends to lead us. I say a different; because the figure itself may lead us to a limited or partial conclusion; the remedy for which I shall explain at length in the next dissertation.

The simplest kind of rectilineal figure is the triangle; the different parts of which the learner ought to be familiarly acquainted with; and to keep in his mind, that besides the triangular space itself; there are six different magnitudes which go to the making it up; viz. three sides and three angles; each of which ought to be particularly attended to. It will be useful also to observe; that the triangles ABG, ACF though they have several parts in common, are nevertheless to be considered as much as distinct and different triangles, as if separated from each other by the distance of a thousand miles.

The

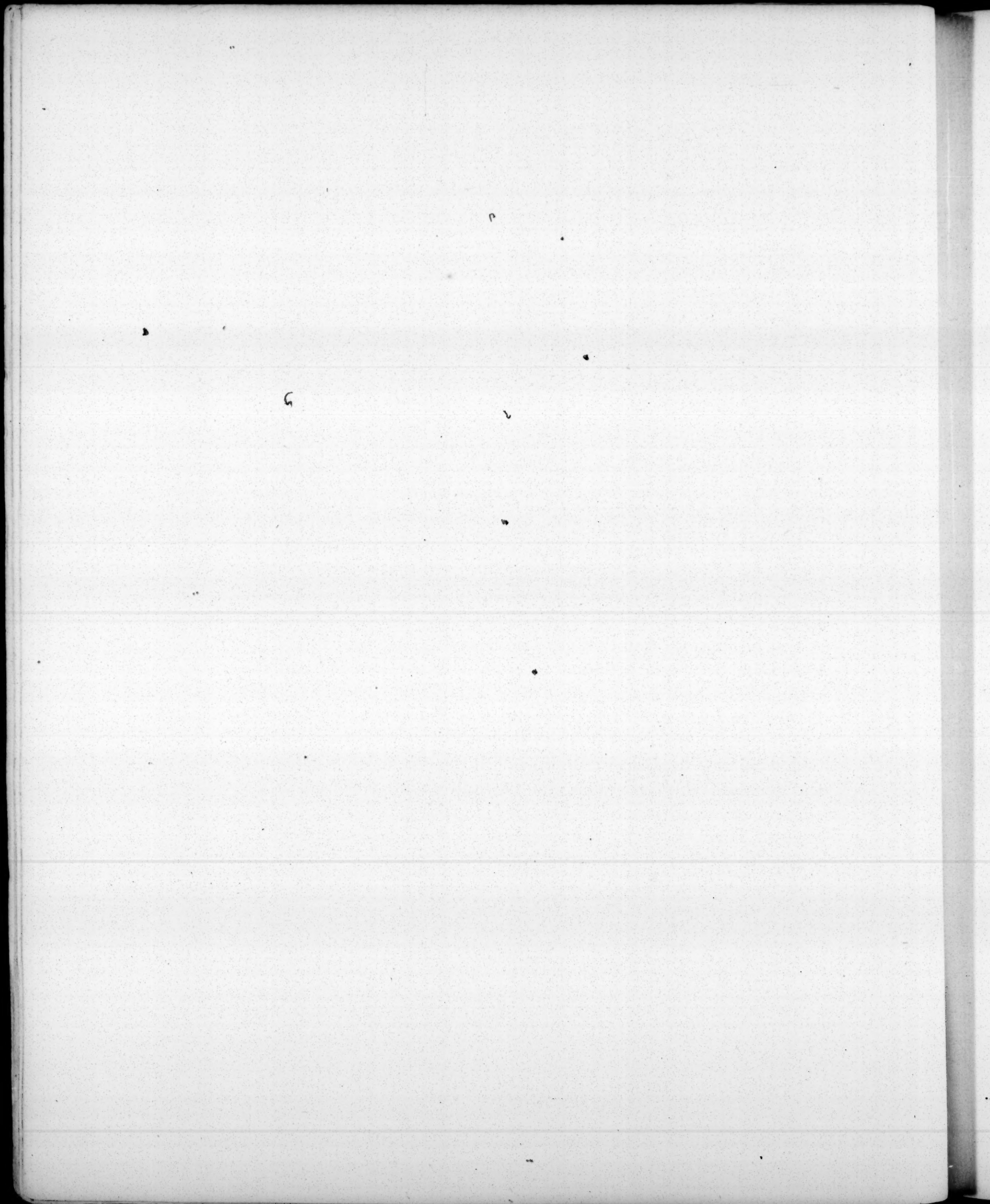
The three sides of the triangle ABG are AG, AB, BG and the three angles are ABG, AGB, GAB: and of the triangle ACF the three sides are AF, AC, CF; and the three angles ACF, AFC, CAF and the angle at A or DAE is said to be common to the two triangles; and therefore they have one angle equal to one angle; and it is said to be contained by the sides FA, AC when it is considered as an angle of the triangle ACF: but when it is considered as an angle of the triangle ABG it is said to be contained by GA, AB.



Again FBC and BCG are also two distinct triangles: and the three sides of the triangle FBC are BF, FC, CB and the three angles BFC, FCB, CBF: and the three sides of the triangle BCG are CG, GB, BC; and the three angles CGB, GBC, BCG: these triangles have a common side viz. the side BC; and this side is said to be extended under the angles BFC and BGC. These things ought to be well understood and strongly impressed upon the memory.

But farther, we find that the two triangles ACF and BFC have a side and an angle common to both, viz. the side FC and the angle AFC which is the angle BFC; also the two triangles ABG, BCG have a side and an angle common to both, viz. the side BG and the angle AGB which is the angle CGB.

I have now explained at some length the original and nature of the geometrical principles; and the instruments made use of for communicating this kind of knowledge, taking notice at the same time of some of the most unusual forms of expression and which are apt to perplex the learner at his first setting out.



DISSERTATION II.

Concerning the nature and extent of geometrical demonstration.

IN the former dissertation, as much care as possible has been taken to avoid the suggesting any circumstance to the reader's imagination, which will not be necessary in the very first steps to be taken in this science. I must therefore beg he will not consider what has been said as words of course, merely to scrape acquaintance, or any attempt to display my own learning, but as seriously intended for his improvement. Nor would I have it thought that this proceeds from any want of inclination to be in the good graces of my reader, much less from want of capacity; for with very considerable deductions both of thought and trouble, like some other commentators I could have flourished away, in an explanation of the eleventh common notion, upon the properties of the asymptotes of the Hyperbola &c; if my intention had been to raise the admiration of the reader, rather than to fix his attention. Having therefore resolved to sacrifice every ornament to this single consideration; the learner, it is to be hoped, will endeavour to convince himself, or rather some more impartial judge, that he understands every thing mentioned in the last dissertation before he proceeds to this; in which he will find the nature and end of geometrical demonstration laid open with all the simplicity and perspicuity which the commentator could give it; and with as much particularity as the patience of the reader, a faculty rather apt to be discomposed by much exercise, could well be supposed to bear.

CHAP.

CHAP. I.

Containing an explanation of the two first propositions.

THE two first propositions are thus explained by Dr. Hooke
 “ Euclid having premised his principles, he begins his method
 “ of demonstration, in which he takes no more for granted than
 “ what he hath already laid down as easy and self evident. His first
 “ proposition then is upon a right line given to make an equilateral
 “ triangle. He hath defined in the fourth definition what he
 “ means by a right line, namely that which lieth straight between
 “ the two extremes of it which are points ; and what he means
 “ by an equilateral triangle, namely, such a one which hath all its
 “ three sides equal to one another.”

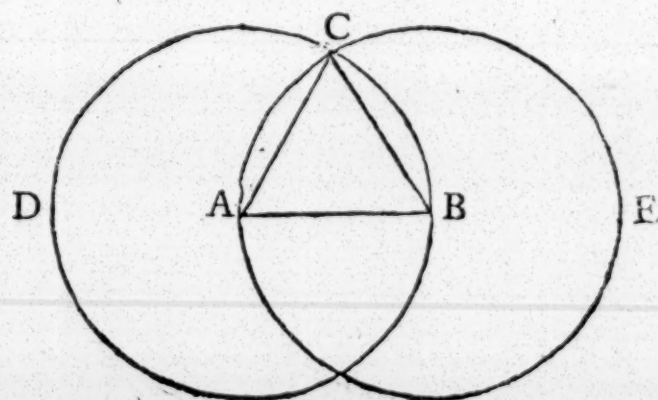
“ This first proposition is a problem, which explains a way how
 “ to do and perform the thing required, as well as shews how to
 “ manifest the truth and certainty of the thing done : It contains
 “ therefore and shews a double invention, without which or some
 “ such other thing, the proposition can neither be done nor demon-
 “ strated; which inventions are called mediums or means by which
 “ we attain to the end propounded or desired. The end here
 “ sought is how from the ends of a line given to draw two other
 “ lines, each equal to the given line, which shall meet in one and
 “ the same point. It is certain that these lines must begin from
 “ the ends of the first line given ; but which of these to draw first,
 “ and which way, and with what inclination to the former line,
 “ that is with what angle, that is not yet known, and some inven-
 “ tion must be thought of, how to direct our ruler to draw it.
 “ Well how shall this be done, since there may be infinite of
 “ lines drawn from each of those points, which shall every one of
 “ them be equal to this line given ? How then shall we among
 “ those infinite or indefinite number chuse out the right ? 'tis im-
 “ possible without some invention. Our author therefore helps
 “ you to one, and one which you have already granted to be feasa-
 “ ble in the third petition. Upon the center A, and distance AB,
 “ draw a circle, says he, BCD ; what then ? To what purpose ;
 Why

“ Why this circle then will give you a line, in which are contained
 “ all the points or ends of the infinite lines, which may be drawn
 “ from the point A any ways that shall be equal to AB. How so?
 “ Why by the fifteenth definition you are taught, that a circle is a
 “ plain figure bounded by or contained within one curve line,
 “ which is called the circumference; to which every right line
 “ drawn from a point in the middle, which is called the center,
 “ are equal to one another. But what are we yet the wiser? How
 “ do we know which of these infinite lines we are to draw? To
 “ which of these infinite points that are in this circumstance? To
 “ know this, you must do the same thing upon the point B: that
 “ is, upon the point B and distance BA, draw or describe the circle
 “ ACE, which will give you all the possible infinite points in that
 “ plain; to the which from the point B right lines may be drawn
 “ equal to BA. Now then since these circles contain all the pos-
 “ sible points of the lines equal to AB or BA that can be drawn
 “ from A or B. It follows that where those circles intersect, there
 “ only must be the point to which those lines may be drawn;
 “ namely at C and at the other point of intersection and no where
 “ else soever. Drawing therefore lines from A and from B to either
 “ of those points as to C as AC, BC; you have done the thing
 “ that was propounded; namely upon the line AB given you have
 “ made an equilateral triangle ABC; which was desired. This is
 “ the first part of the pro-

“ blem, and indeed the
 “ difficultest to find out;
 “ namely how to do the
 “ thing required; and in
 “ this part lie the greatest
 “ difficulties of mathema-
 “ tical knowledge, to wit,
 “ in finding out the pro-
 “ per and true mediums

“ or means to perform the problems required to be done, which
 “ for the most part are of the same nature with this, and con-
 “ sist in the finding out the position of a point: for this pro-
 “ blem might have been thus worded. A right line being given

“ as



“ as AB, to find a point as C, to which lines being drawn from
“ the points A and B, they shall each of them be equal to one
“ another, and to the line AB which is given : or two points A
“ and B being given to find a third point as C, which shall have
“ the same distance from A and from B that they have from one
“ another. But our author not having given any definition of dis-
“ tance or equality, otherwise than may be collected from equality
“ of the sides of some figure ; or of the rays or lines drawn from
“ the center to the circumference of a circle, he chuses rather to
“ make use of an equilateral triangle to find out that propriety of
“ a point so posited.”

“ The second part then of the proposition is the demonstrative
“ part thereof, namely, to prove from the principles already laid
“ down and granted and assented unto for true and certain, by a
“ clear chain of reasoning and deduction, that these lines AC and
“ BC are each of them equal to AB and so equal to one another ;
“ and consequently that the figure ABC bounded and limited by
“ them is an equilateral triangle, according to the description of
“ that figure in the twenty fourth definition. The next thing then
“ to be invented or found out is the medium or means of demon-
“ strating it to be such ; for this we have two. First, the definition
“ of the properties of the lines from the center of a circle to the
“ circumference in the fifteenth definition, that they are all equal
“ to one another. And secondly we have the first axiome, those
“ which are equal to one other are equal to one another. First AB
“ is equal to AC, because they are right lines drawn from the center
“ A to the circumference BCD ; for by the fifteenth definition, as
“ I said, all such lines must be equal. Next BA and BC must
“ upon the same account be equal to one another, because they
“ also are lines drawn from the center B to the circumference
“ ACE : Therefore both AC and BC are equal to AB ; but by the
“ first axiome those which are equal to one other are equal to one
“ another, therefore AC and BC are also equal to one another.
“ Therefore the three sides of the figure ABC ; namely, AB, AC,
“ and BC are equal to one another, and consequently bound an
“ equilateral triangle according to the twenty fourth definition,
“ therefore

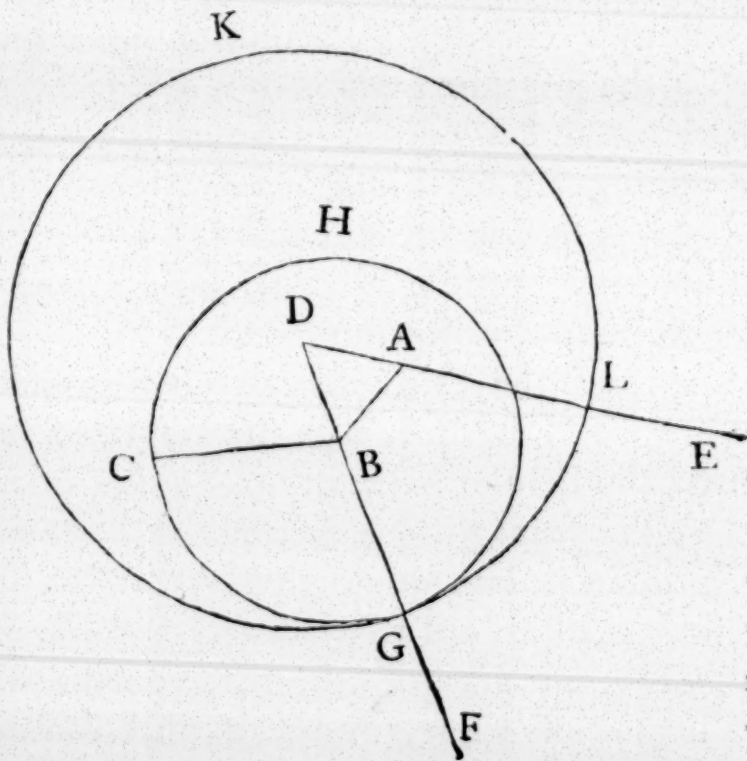
“ therefore upon the line AB given, an equilateral triangle is made,
“ which was the thing to be done and proved.”

“ THE second proposition is also a problem.

“ From a point given to draw a right line, which shall be equal
“ to another right line given. For the performing of this propo-
“ sition, it being a problem, there is need of two mediums to be
“ invented or found out; the first is for the doing of the thing re-
“ quired; and the second for the demonstration of it; and both
“ these are to be fetcht out of the principles already agreed to, or
“ from the truth evidenced in the preceding proposition: For we
“ are not to suppose any thing further known in this science, and
“ therefore much less are we to make use of it. Searching there-
“ fore our store, we have no other medium to make a line equal to
“ a line, than first by the help of a circle, defined definition 15.
“ which by the third *postulatum* is granted to be describable upon
“ any center and at any distance: Or secondly, by an equilateral
“ triangle described by definition 24, which we learnt how to make
“ by the preceding problem: For as to the equal sides of an isos-
“ cles triangle, definition 25, or the equal sides of a square, a pa-
“ rallelogram or oblong square, Rhombus or Rhomboides descri-
“ bed definition the 30, 31, 32, and 33. Though their properties
“ are there defined, yet we are not taught how to make them as,
“ yet, and consequently can make no use of them, as *media* to
“ perform the thing required to be done by this problem. Nor
“ are we to suppose, that the length of the line given may be taken
“ by the help of a measure or a pair of compasses, and transferred
“ to the point given; because those are first not mentioned in the
“ principles laid down there, which you are to make use of, and
“ of no other, till they be accepted for principles undeniable: For
“ this is not yet granted, that you can with compasses, take a true
“ length of a line, much less that you can transfer it and set it in
“ another place. But you have granted that 'tis possible, upon a
“ point given at any distance, to describe a circle, or suppose it so
“ done, which is sufficient for the demonstration, that being the
“ principal thing aimed at by our author, namely, to lay open to
“ the understanding the reasons and grounds of the properties of
“ quantities so and so qualified, that you may plainly see how and

“ for what cause things are thus or thus and cannot be otherwise.
 “ For as to the most practicable and expedite ways of doing those
 “ things mechanically, and for other uses, *that* belongs to another
 “ part of mathematics ; namely to the practical part thereof, which
 “ is called practical geometry, or mechanical geometry, which
 “ ought not to be learned till this be first known ; but this which
 “ our author treats of is speculative geometry, and principally aims
 “ at demonstrations or explaining the proprieties of quantities to
 “ the understanding. You saw clearly by the former proposition
 “ why ABC was an equilateral triangle ; and there could be but
 “ two such made upon the line AB in the same plain ; there being
 “ but two points where in those circles cut each other ; those cir-
 “ cles determining all the lines equal to AB that can be drawn
 “ from the points A and B. His way then of performing this
 “ problem is this : Let the right line given be BC and the point
 “ given be A ; from
 “ which point a right
 “ line is to be drawn
 “ equal to the line BC.
 “ First draw a line from
 “ B to A, which is
 “ granted possible by
 “ the first postulatum ;
 “ then by the former
 “ proposition upon this
 “ line, BA make an equi-
 “ lateral triangle BAD :
 “ then upon the center
 “ B and distance BC
 “ describe a circle, as
 “ CGH by the third
 “ postulatum ; then by
 “ the second postulatum produce DB to F ; then upon the center
 “ D and distance DG draw the circle GKL, then as before, pro-
 “ duce the line DA to E ; there shall AL be the line required to
 “ be drawn from the point A equal to the given line BC.”

“ This



“ This is the construction of the problem, or the preparing of
 “ the proposition fit for demonstration, by which you may clearly
 “ understand the reasons of it, deduced from the few principles
 “ already laid down : For first, that BC is equal to BG is clear
 “ from the fifteenth definition, which determines the propriety of
 “ equality of the rays of a circle. Next that DBG is equal to
 “ DAL, is as clear from the same definition, they being both rays
 “ or lines drawn from the center D to the circumference GKL by
 “ the construction premised. Thirdly that DB is equal to DA, is
 “ clear from the construction ; for DBA is an equilateral triangle,
 “ two of whose sides DB and DA are. Now by the third axiom
 “ or common notion, if from equal quantities you take equal quan-
 “ tities, the remainders shall be equal ; if from DL you take DA ;
 “ and from DG, DB ; the remainder AL shall be equal to the re-
 “ mainder BG ; but BC is also by the construction equal to BG ;
 “ therefore since by the first axiom these two quantities which are
 “ equal to one other quantity are equal to one another ; therefore
 “ BC and AL, being equal to BG, are equal also to one another ;
 “ therefore from the point A the line AL is drawn equal to the
 “ line BC ; which was the thing to be done and proved.”

“ Now though this way of demonstration and reasoning may
 “ seem tedious and too long to detain the mind and attention in
 “ the finding out the proprieties of quantities, yet 'twas the way
 “ made use of by the ancients. And 'tis altogether necessary, espe-
 “ cially in the beginning of this study, to accustom the mind to
 “ attention and circumspection, that it may receive nothing for
 “ truth but what it sees clearly by the reasons and causes of it,
 “ that thereby the mind may acquire an habit of intention, and
 “ of examining the whole chain of consequents from the first prin-
 “ ciples to the truth evidenced. For the want of which, some
 “ small error perhaps may slip into the mind under the appearance
 “ of truth, and thereby make all the subsequent reasonings and
 “ deductions unsound ; and 'tis very much harder to clear and free
 “ the mind from it when once received, than to prevent the recep-
 “ tion thereof. There cannot therefore (in this study especially,
 “ not now to mention any other, where it is possible it may be
 “ altogether as convenient, nay necessary) there cannot I say

“ therefore be as I conceive to much circumspection and caution
 “ used in admitting principles, and furnishing the mind with the
 “ true grounds of knowledge ; because for the most part we are too
 “ prone to take up every thing we hear upon trust, without exami-
 “ nation : we are too apt to run away with a thing, and think we
 “ know it and see it clearly before we are sure we do ; and are
 “ impatient of delay in examining and considering ; whereas if the
 “ mind be a little at first accustomed to this leisurely and strict
 “ way of reasoning, after it has got a habit it will make as much
 “ dispatch in receiving things with sufficient examination, as ano-
 “ ther shall without it. And the patience only is needful for the
 “ most part at first to beget attention ; nor is it peculiar to this
 “ acquisition alone ; but we see it necessary, and practised in many
 “ other things where a good habit is to be acquired ; as in reading,
 “ writing, music, drawing, and most other manual operations.
 “ The roots and beginnings of knowledge, and practice too, are
 “ bitter and tedious, but the fruits are sweet and pleasant ; and
 “ whosoever attains the end, will never repent the time spent in
 “ the beginning.”

C H A P. II.

The same subject continued.

IN the preceding chapter I have given Dr. Hooke's explana-
 tion of the two first propositions, in order to shew the reader that
 I am not singular in my opinion, that Euclid is not to be under-
 stood, unless the learner bestows that attention upon the first prin-
 ciples, which may enable him to carry along with him their full
 meaning and import. It is true that some confused facts concerning
 the properties of figures may be picked up by a very careless perusal
 of this author : but whoever is satisfied with this, had better look
 for his knowledge some where else ; because he will meet with a
 great many impertinent interruptions to his scheme from the several
 steps of the demonstration ; and rather content himself with a
 kind of law evidence, by resolving to consent to every proposition
 which is delivered as truth by two or more credible authors.

Although

Although I approve in general of what Dr. Hooke has advanced, yet I am convinced that a more minute consideration of the different parts of each proposition will be necessary for the student who would wish to leave his prejudices behind him as he advances. I shall therefore point out these circumstances, which he might be apt either to overlook or mistake. Supposing the first proposition of Euclid carefully examined, he will find it taken for granted that the two circles cut one another, and that this supposes the circles to be described upon the same even surface, or according to the geometrical language in the same plane, otherwise there could be no such point as C. Again the supposing C to be a point implies that the two lines, the two circumferences of the circles have no breadth, otherwise the point would have parts. Likewise by supposing the triangle ABC to be an assignable or determinate magnitude implies that the points A, B, C have no magnitude for if they had parts various triangles differing both in sides and angles might be described, and it would be impossible ever to arrive at a determinate conclusion.

But farther, the reader is to consider that he has been reasoning all this while, upon a particular straight line of very inconsiderable length; He is therefore next to inquire, whether his constructions and conclusions are also particular. That they are not will appear from this, that no consequence is supposed to follow from the straight lines having any particular length, nor is any construction undertaken upon that supposition; only that the line be finite that is, that we know its two ends.

But it happens rather unluckily, that though the scientific construction be general we are nevertheless forced to take up with a particular one; which is very apt to create prejudices unless our attention be every now and then called to the general construction: we must remember that, though we work with a ruler and compasses, the science knows no such instruments. Let us suppose the line AB ten miles in length; and that the same construction is to be performed; we shall now get beyond the objects of our senses; but if we have bestowed the necessary attention to the problem, the understanding will have as clear and distinct a perception as it had before, when the line was perhaps not above three inches long.

Our

Our author's plan obliges him strictly to prohibit the use of any particular instruments; straight lines and circles are supposed to be drawn and described by mechanical operations; and we are left to guess in what manner. It will however be prudent to acquire some one ready mechanic way at least of constructing every problem. For I always suspect any pretensions to general truths which have not been collected from an examination of particulars. The ruler and compasses are such exact minature representations of the two postulates, that they are to be preferred to every other instrument for assisting the imagination to follow Euclid's constructions: only the compasses are to be confined to their proper use, always to describe circles, but never to measure distances. Indeed it is not always necessary, and often would introduce confusion if the circles were compleatly described: it is therefore sufficient to describe as much of them as is necessary; as for instance, in the future application of this proposition, it is obvious that it will be sufficient to find the point C; because the point where the circumferences of the circles cut one another gives us every thing necessary for the construction.

We come now to the second problem which is of a more complicated nature than the first; and yet there are very few, upon reading it, who do not fall in with the notion that Euclid has taken a great deal of unnecessary pains to do what a single opening of the compasses would have performed with equal or perhaps more exactness. But considering the science *only* with a view to practice, the affairs of mankind are carried on upon a much larger scale than to be managed with a ruler and compasses; and we have only to suppose the line BC a mile in length, and this will answer the objection sufficiently. The understanding is capable of reaching the general conclusion delivered in this proposition; why then should it be fettered with instruments? Euclid's contrivance is admirable and suited to the dignity of the human mind. According to his plan we are guided by instruments, but neither confined nor loaded with them.

After the student has fixed in his memory the construction and demonstration of this second proposition, it is more than probable that his conclusions will be limited to the particular figure which
he

he made use of and to that very position which the lines happened accidentally to take.

This is a very general prejudice, and will be removed by attending to the following directions. Let the learner consider whether DA and DB produced will cut the circle CGH : and he will find that DB must cut it as it passes through its center, but that DA's cutting it depends upon the length of AB. Change the position of the point A and repeat the construction and thus you will learn what lines have a fixt position or only an accidental one : describe the equilateral triangle ABD upon the other side of the line AB and compleat the figure ; also join the points A and C instead of the points A and B and repeat the same constructions ; and by proceeding thus you will be able to conquer the prejudice of sense and to acquire something like a scientific view of the problem. For you will find no consequence deduced from BC's having any particular length or the point A any particular position.

But the third proposition will afford an opportunity of taking a somewhat different view of these two propositions. It is a problem. Two unequal straight lines being given to cut of a part from the greater equal to the less. The simplest case of this proposition would be, when both the lines are drawn from the same point : for making that point the center, and the shorter line the radius, the circle so described would solve the problem : but if the lines had any other position, a very different apparatus would be necessary ; we must learn how to put a line at the extremity of the longer equal to the shorter line ; and this cannot be done before we have learned how to describe an equilateral triangle. I would therefore propose it as an exercise for the student, and his success would be a proof that he was master of the subject, to set out upon a supposition that the book began with this third proposition, which will now include the first and second ; and by this process he will have their connexion with each other strongly impressed upon his mind. His figure will consist of five circles, and their use is to determine four points : such a point as C in the first proposition, G and L in the second ; and E in the third.

Whoever attempts to communicate knowledge to mankind, must write upon the supposition of a certain degree of improvement
in

in those to whom he addresses himself, otherwise his book can have no determinate end to answer. Euclid supposes his reader above the prejudice of sense, and to have the ready use of his understanding, with a due command of attention; and upon this supposition he has said every thing necessary to convey the fullest information to his reader. But I write to a different sort of people, to such as are immersed in the prejudices of sense, and at the same time very thoughtless; to those whose understandings are not disposed to attend to the call of reason without frequent admonitions; and this is my apology for begging the student's attention to one thing more, before I finish this chapter; he will always find something supposed or given in every proposition, and perhaps nothing will contribute so much to a right understanding of the proposition, as a diligent enquiry into the use made of the *data* or supposition, as for instance whether the reasoning in the third proposition, does not depend upon the supposition that AB is longer than C ; and where the reasoning would fail if that were not the case. Our author was very confident that his reader would know, that, without this part of the supposition, there never could have been such a point as E : but I am afraid that mine trusted to his senses for the real existence of this very point; dont I see, says he, that the circle cuts it.

I shall conclude this chapter by reminding the reader again, that the use of the two circles in the first proposition is to find the point C : and in the second, that the equilateral triangle is described to fix the point D , which is to be the center of a future circle; whose radius DG is determined by the description of the circle CGH ; and lastly that this circle itself is described to find the point L . And thus I leave the student to pursue his own meditations upon these three propositions, only advertizing him, that, if he thought them easy upon the first reading, and still persists in the opinion that he then understood them, I am certain he knows nothing of the matter.

C H A P. III.

Concerning hypotheses.

THE fourth proposition is of a different kind from the three first ; and is called a theorem. In the problems something is required to be done : a problem therefore may be divided into two parts the construction and demonstration : for after shewing in what manner it may be done ; it is necessary to prove that this is agreeable to those principles which the nature of your science obliges you to adhere to. Thus in the third proposition it is totally inconsistent with the principles of this science to take the line C in a pair of compasses ; and by this means cut off AE from AB ; for whatever instruments we use, it can be allowed to do nothing more by their assistance than to draw a straight line and describe a circle ; the compasses then may be used in describing a circle ; but it can do nothing else without encreasing the number of our mechanical problems, which Euclid has avoided with the greatest care ; and here it is to no purpose to alledge that it can perform the one as accurately as the other : but I desire it may be observed that it is not admitted upon the score of its accuracy, but as keeping in some respect to the idea of the postulate ; for its merit in point of accuracy is really nothing.

If geometrical knowledge could be communicated in the form of problems it would make a readier and stronger impression upon the mind of the learner : for in the construction of a problem the student is not barely a spectator but has an active part assigned him, to keep up his attention : but when a theorem is presented to his consideration, unless he has learned the use and management of an hypothesis, he will be but an indolent and inattentive spectator ; and to tell such an one that the consequence alluded to follows from the supposition is to address him in a language which he does not understand. But for a more particular explanation I shall analyse the fourth proposition, which runs thus ;

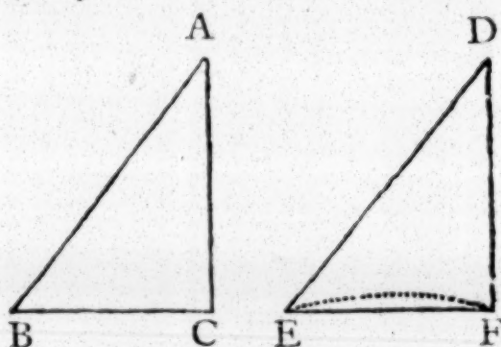
If two triangles have the two sides equal to the two sides, each to each ; and have the angle equal to the angle, the *angle*, contain-

ned by the equal straight lines : They will also have the base equal to the base ; and the triangle will be equal to the triangle ; and the remaining angles will be equal to the remaining angles, under which the equal sides are extended, each to each.

This general enunciation of the proposition will convey but a very indistinct perception of its meaning to a learner, especially upon a first reading ; this however may be improved and rendered distinct by the assistance of a figure, representing two triangles in the supposed circumstances : but here again a new difficulty arises, as this hardly ever fails to bring a prejudice along with it : for the triangles representing not only the suppositions, but at the same time the inferences which are said to follow from them ; the student is at a loss to distinguish what is given from that which is to be inferred from it ; because in all probability he will look upon the figure itself as the only source from which his knowledge is to be derived ; and then he is as well convinced that the conclusions are true as the suppositions ; and cannot conceive what it is he has to demonstrate. And unless he be qualified to lay a proper stress upon the supposition, the demonstration must appear to him an idle abuse of words calculated only to perplex his understanding, especially if he has already fixt his opinion by his compasses and other instruments. It is not easy to devise a ready remedy for this error ; so that a man runs a risque either of having no opinion concerning what is proposed in the proposition or an absurd one ; for of all the difficulties attending the acquisition of the science, this is the hardest to be got over. To reason accurately from a supposition is no easy matter ; attention and habit will do a great deal, but above all a proper sense of the difficulty of it. In the problems which we have already considered, the student has only to attend to the works of his own hands, in the first proposition the straight line AB is given him ; but the two circles, and the two straight lines AC and BC are his own manufacture ; and likewise in the second so is the whole figure except the point A, and the straight line BC : this makes a distinction which nobody is so thoughtless as to overlook. But in a theorem like this fourth proposition ; where one has nothing to *do* but to *think*, the case is very different, especially if the figure, which is intended to direct him,

him, by mistaking its use, should become the occasion of several prejudices. But to be more particular,

Let ABC and DEF be two triangles; having the two sides AB and AC equal to the two sides DE and DF; each to each, viz. the side AB equal to DE; and the side AC equal to DF; and the angle contained by BAC equal to the angle contained by EDF.



This now is the hypothesis or supposition, which is to be considered as absolutely certain; and a principle from which we are to reason as confidently, as from the first common notion that magnitudes, equal to the same, are equal to one another. Whatever might be collected from looking at the triangles, or by any other means, is to be entirely neglected; and not merely neglected, but even shunned as leading to the most pernicious errors. The student has only to examine the hypothesis, one part after another, making use of the figure only to assist him in comprehending its meaning, which he might set about in some such manner as this, repeating to himself, the two sides I find are equal, but not any how; for they are said to be equal each to each; the sides of all triangles are inclined to one another; but here a particular inclination is specified; they are said to be equally inclined: this the attentive reader will comprehend perfectly, and be able to say I understand now what the author means; he affirms that all triangles which agree with one another in the circumstances above mentioned cannot possibly fail in giving us the same consequences. But let us see what consequences he says will follow from his suppositions—They will also have *the third side or* base BC equal to the base EF; and the whole triangle ABC equal to the whole triangle DEF: and the remaining angles — *stop*, why that epithet remaining? Why, because only one angle in each triangle being equal by the supposition, are there not then two in each triangle which remain to be considered? But to proceed, the remaining angles shall be equal to the remaining angles not both taken together, but each

to each. EACH TO EACH, this though somewhat particular nevertheless admits of a latitude sufficient to perplex an ordinary understanding; and without some additional circumstance, must remain unintelligible even to the acutest. But the angles are particularly described and also fixed; because they are to have the lines which are equal by the supposition extended under them; that is the angles under which AB and DE are extended are to be equal, as also those under which AC and DF are extended; viz. the angle ACB or BCA equal to the angle DFE or EFD; and the angle ABC or CBA equal to DEF or FED. Now these are said to be the consequences which will certainly follow from the forementioned suppositions.

But it will be too much for the learner to attempt the whole demonstration at once; let us therefore see what consequence will follow from each part of the supposition taken singly. The principle by which they are to be examined is the eighth common notion, magnitudes which apply themselves exactly to one another are equal. It is here said that AB and DE are equal. What do you understand by this word *equal*? Certainly it must mean, that, if the straight lines AB and DE be properly placed, they will apply themselves exactly to one another. But what is the proper way of placing them? Put the point A upon the point D and the line AB upon DE; observe all this may be done whether the lines be equal or unequal; but then the point B will apply itself to the point E only when the lines are equal; for when AB is longer than DE the point B will be found beyond E; and when AB is shorter than DE the point B will be found between D and E. After a careful examination of these two particular lines; let us suppose each of them a thousand miles in length; here our senses forsake us, but surely our understanding unless it be weak indeed, will give us as positive a conclusion as in the other case; so that we may conclude universally that whatever be the length of AB provided only AB and DE be equal; and the point A upon the point D, and the line AB upon DE; the point B will always apply itself to the point E; which was the first consequence to be examined. Also if the point A be upon D and the line AC upon DF; the point C will apply itself to the point F, as is obvious
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for the same reasons; and that will always be the case whatever be the length of AC.

We come next to consider in what circumstances it is possible to place AB upon DE; and at the same time AC upon DF: Now this can always be done, if the angle BAC be equal to the angle EDF; because when the point A is upon the point D: and AB upon DE; it follows from the equality of the angles that AC will take the direction of DF. But if the angles be unequal, although the point A be upon the point D; and AB upon DE; yet AC cannot possibly fall upon DF; because AC will then take a direction, on the one side or the other of DF, according as the angle BAC is greater or less than EDF.

Lastly let us suppose the two extremities of two straight lines to be the same; may we not conclude from this that the lines are equal, and in the same direction, that is the one line will apply itself to the other; because if it did not the two straight lines would inclose a space. Therefore in this figure of ours, if it can be proved that the point B applies itself to E, at the same time that the point C applies itself to F; we may certainly conclude that BC will apply itself to EF and be equal to it.

Each of these suppositions ought to be examined frequently, and the consequences which follow from them are to be strongly impressed upon the memory; so that the very mention of the supposition may suggest the consequence.

And here it may be proper to inform the student, that he is not to consider this as any acquisition of scientific knowledge; but only the consequences which common good sense cannot fail to draw, whenever these things become the subject of consideration; these remarks are introduced not as teaching any thing new, but only to fix the attention. However the combination of these, as in the fourth proposition, will carry us a step beyond the common sense of mankind: for he who first found out that the two sides and included angle fix every part of the triangle beyond a possibility of change, had certainly more than a common notion of a triangle, and merited the high title of an *inventor*. The learner is now to endeavour to make himself master of the fourth proposition, before he proceeds to the next chapter, in which he will find

find some remarks tending to correct, confirm or enlarge the notions which he may derive from it according to the different degrees of attention bestowed upon it.

CHAP. IV.

The same subject continued.

THE reader is without doubt surprized to find this affair of *suppositions* represented as a difficulty almost unsurmountable; and not a little inquisitive, we may suppose, after the reasons of this strange *phænomenon*. Our own indolence is the real cause, for simple reasoning is hardly sufficient to set the mind in motion without some external application. When a problem is proposed, there is something to be done, and, which is very much to our purpose, may be done wrong: this sets us upon thinking; and by giving a preference to one method above another, we come to distinguish between what is right and wrong, and become so much interested as to give the question a serious examination. But the case is very different in theorems, for the consequences are absolutely fixt by the suppositions, so that a mind without experience of such subjects has nothing to engage its attention.

It would be absurd to propose a problem of this kind: Suppose two triangles to have two sides equal to two sides, each to each; and the angles, contained by the equal sides, equal; it is required to make the third side, equal to the third side; and the two remaining angles equal to the two remaining angles, each to each; namely those under which the equal sides are extended. Here there is no room for a construction; because according to the supposition, the sides and angles cannot be otherwise than equal. However it is necessary to shew that these magnitudes are really equal; and the supposition alone is what we must trust to, for bringing this about: and we are at a loss because, for this end, we begin to derive the consequences from the supposition before we have examined the different circumstances of which it consists, or without understanding what is implied in the supposition, when it reaches beyond the obvious meaning of the words in which it is expressed;

expressed ; or when *more is meant than directly meets the ear*. The mind must have room to exercise itself in, as well as the body ; and it is rendered almost inactive by being confined to a single supposition. But if the student has patience this may be remedied by varying the supposition ; and then you may have all the latitude and range of thought, which is allowed in the construction of a problem.

Let us suppose then that only AB is equal to DE, and the angle BAC equal to the angle EDF ; but that we are quite in the dark as to AC and DF.

When the point A is put upon D, and AB upon DE the point B will apply itself to E as before and AC will take the direction of DF ; but it is impossible to determine any thing concerning the position of the point C, only that it will be found some where in the line DF or in DF produced.

Again let us suppose only the two sides equal ; then the point A being put upon D ; and AB upon DE ; the point B will as before apply itself to E, but we can make no use of the other part of the supposition ; for we can determine nothing concerning the direction which AC may take, because nothing is said concerning the angles or inclinations of the lines.

But farther, let us suppose, that not only AB is equal to DE and AC to DF and the angle BAC equal to the angle EDF ; but moreover, that AB is equal to AC and consequently DE to DF : now by placing the triangles as in the proposition, we shall find the angle ABC equal to DEF ; and not only that, but, by putting A upon D, and AB upon DF the point B will now apply itself to F because AB and DF are equal by the supposition ; and also the angle ABC will be equal to the angle DFE : but ABC has been already proved equal to DEF ; therefore by the first common notion the angle DEF is equal to DFE : that is the angles at the base of the triangle DEF are equal : But this conclusion does not depend entirely upon DE's being equal to DF ; but it is required besides that BA should be equal to AC and to each of the lines DE, DF and likewise the angle BAC equal to EDF. So that this case can by no means be considered as including the fifth proposition, in which there is no other supposition but the equality

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of the two sides ; and as we must have an angle equal to the angle at the vertex of the isosceles triangle, this angle must be made, which, in our present circumstances, will not be found such an easy matter. And so much for hypotheses or suppositions.

CHAP. V.

Containing a critical examination of the fourth proposition.

OBJECTIONS are brought against this proposition, as if the demonstration proceeded by a mechanical application of the two triangles to one another. But whoever starts such an objection as this, has not the most distant conception of the demonstration ; or else he must be ignorant of the nature of a mechanical application. And a particular examination will put this beyond all doubt.

Suppose two triangles of brass, made as accurately as possible, all their different parts corresponding exactly. When we apply them to one another, all that we can say is, that as far as our senses can judge, the parts seem to agree. Now what knowledge is got from this ? Certainly by this application, no property of a triangle can be discovered ; we may form a conclusion concerning the accuracy and neatness of the workmanship ; but nothing farther. It is impossible to say from this that any thing is equal, but what is made equal ; no doubt a very curious discovery and tending greatly to the enlarging the boundaries of the science !

The mistake here arises from not considering that it is impossible to make the parts of the figure which are to represent the supposition unless you make at the same time the parts expressing the consequences which, it is said, will follow. The parts representing the supposition are to be tried and examined by the eye, as well as those which represent the consequences ; so that if any consequence follow it must follow from nothing. And the most attentive student will retire from this contemplation, with a very curious piece of information ; namely, that he had seen two brass figures so contrived as to fit each other exactly.

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We now proceed to consider the application made use of by Euclid in this fourth proposition. And first it is to be observed that his triangles are required to have no particular position; therefore the point A may be supposed to be upon D as well as any where else; and likewise the line AB upon DE; and all this might happen to any two triangles not determined to any particular position. Then AB being to DE the point B must apply itself to E, it is impossible to conceive it to have another situation, which is very different from saying that as far as we can judge by our senses it seems to be so; again if the angle BAC be equal to EDF, the line AC must take the direction of DF; and it is impossible to conceive it to take any other; and so likewise the point C must apply itself to F, whenever the line AC is equal to DF: and so on to the coincidences of the different parts of the triangles; and all this is as different from a mechanical application, as light is from darkness. For supposing what is taken for granted, it is impossible to conceive the consequences not to follow; and this is certainly science if there be any such thing in the world.

But again it is alledged, why may not this method of applying be extended still farther by making it an instrument for constructing problems? as for instance in the second proposition; why may we not suppose the line BC so placed that B may be upon A, and the thing required is done? For a very good reason, because BC and the point A have each of them a fixt position already.

Upon the whole then we may conclude, that this method of application is perfectly scientific; and that whenever two triangles agree in the circumstances mentioned in this supposition; the consequences will always necessarily follow; the bases will be equal and the remaining angles, each to each, viz. those under which the equal sides are extended: Because no consequence is deduced from the lines having any particular length, but only from their being equal: nor is it supposed that the angle contained by the sides is any particular angle, but only that the angles are equal. And this may suffice for an answer to the objections commonly brought against the demonstration of this proposition.

CHAP. VI.

Containing an explanation of the fifth proposition.

ONCE upon a time a certain father resolving not to be imposed upon by reports, determined to examine into his son's progress in this science, produced the book and required him to demonstrate a proposition to which he referred: the young man though unacquainted with the subject, taking courage from his father's ignorance, began very impudently in some such manner as follows; Because the angle ABC is equal to the angle CBA, therefore the angle DEF is equal to the angle CEF &c &c; ringing the changes upon sides and angles, until he had spun out his demonstration to a decent length: and then kept silence in expectation of his father's opinion; who with a grave and important countenance remarked, "This is what we call demonstration."

Every one is sensible, that it is contrary to common sense to imagine that the letters of the alphabet thus repeated can have any meaning; but the indolent reader ought to be informed, that the repeating such phrases, in a regular order, mends the matter but very little, unless they convey to the mind their proper meaning: and unless the fourth proposition be well understood, the most of that which follows will be nothing but an insignificant jargon. As the whole science therefore depends so much upon an accurate and comprehensive view of this proposition, it would be proper for the learner, before he proceeds farther to take the opinion of some acquaintance skilled in these matters; who, by a particular examination might be able to determine, how far he can be properly said to understand it; and this friend is authorized, upon his failure in any point, to admonish him, by saying, "this is what we call demonstration."

I myself can trace every mistake concerning the following propositions, or partial conception of their meaning, up to my ignorance of the full import and meaning of this proposition. For it is by no means to be understood as applicable only to such triangles, as one may make use of to assist the imagination in tracing the

the steps of the demonstration ; but as carrying with it this extensive and general meaning, that all triangles which have two sides equal to two sides, each to each, and the angles, contained by those sides, equal ; that all such triangles, I say, have their bases equal ; and the two remaining angles in every triangle, equal, each to each ; viz. those angles under which the equal sides are extended. Or more properly that all such triangles, whatever be their number, are but one and the same triangle. Or otherwise that the two sides, and the angle contained between them, fixes every part of the triangle, beyond a possibility of change. But I shall now proceed to consider the fifth proposition. And the reader may recollect an attempt in the end of the fourth chapter, to derive this proposition from the fourth, or rather to make it only a particular case of that proposition, when the two sides of each triangle are equal to one another ; and I there gave the reasons why it could not be considered as such : nevertheless the artifice of this proposition will be the better understood by prosecuting that scheme a little farther, producing the equal sides, and proving the angles below the bases of such triangles equal, in the same manner as the equality of those above the base was there demonstrated.

It was then observed that the supposition was too complicated for this purpose ; because not only DE and DF were to be equal to each other and to AB and AC, for this might have been allowed ; but also the angles between the sides were to be equal, which could by no means be allowed ; because according to the fifth, only one angle being given, the other was to be made equal to it ; which is not at present possible.

If the reader has prosecuted this speculation far enough, he will certainly admire the ingenuity of our author for his contrivance to make these angles equal, in a very elegant manner indeed : and which, it is curious to observe, necessarily requires that the sides of his two triangles should be unequal. For he makes this very angle which we are considering common to both, by making the sides of the triangles take the same direction, the shorter side of the one being upon the longer side of the other.

But now it will be proper to turn to the demonstration itself ; and after a careful examination of its different parts, I would re-

commend a particular attention to the use which might be made of it, in order to impress the last proposition more strongly upon the memory. The great advantage of problems above theorems, for fixing the attention of the learner, has been mentioned already: and here a little of that advantage may be gained, by the particular view of the fourth exhibited in this proposition; which will teach us how to represent by an actual construction, the suppositions in the last proposition: because, if we draw two undetermined straight lines, making any angle; we can cut off AB equal to AC; and AF to AG; and, by joining BG and FC form two triangles, with all the parts of the supposition in the last, accurately represented; and this not in imagination, but constructed by ourselves; and the consequences, not made, as was objected to the mechanical triangles, but left to follow from the construction; the accuracy of which, the flow of imagination may examine by instruments; and thus get a kind of palpable evidence for the truth of the proposition; and by varying the inclination of the lines, and the length of the sides, climb up by degrees to something like a scientific conception of its meaning.

The other two triangles FBC and CGB, though they represent a particular instance of the fourth proposition, are not so fit for this purpose, not being so general as the first two triangles nor so much in our power, or rather indeed they are not at all in our power: We may vary the angle and the two sides in the triangle ABG at pleasure; but we have no power over the sides, or angle BFC of the triangle FBC; for they become such as may happen from the construction of the first triangle: therefore the epithet *any* could not be so properly applied to the triangle FBC as to ABG.

I have recommended to those whose imaginations are slow or inactive, these constructions and instrumental proofs of the several conclusions which follow from the suppositions in the fourth proposition. And I now, rather more earnestly, recommend the same to those whose imaginations are disposed to out run their judgement, lest they should snatch the conclusions without the premises.

I have only one remark more to make upon this proposition, which the judicious reader has already made for himself, namely, how little there is to attend to, in this very formidable proposition,
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this *pons asinorum*, upon a supposition that the fourth is perfectly understood: for except the circumstances of taking the equal angles CBG, BCF from the equal angles ABG, ACF; it has nothing peculiar to itself; every thing else consisting of particular prospects of the fourth proposition.

But if any one is desirous to have some kind of reason for fixing his attention a little longer upon this proposition, by taking a somewhat different view of the subject; let him suppose, instead of this fifth proposition, it is required to demonstrate that all the angles of an equilateral triangle are equal; for instance suppose the angles of the equilateral triangle ACB described in the first proposition. Produce CA, and CB; and the same construction and demonstration made use of in this fifth, will prove that the angle CAB is equal to CBA; and by producing AB and AC; in the same manner it may be proved that the angle ABC is equal to ACB; but it has been already proved that ABC is equal to CAB therefore, by the first common notion, the angle CAB is equal to ACB &c.

I shall conclude this chapter by desiring the reader to take every opportunity of correcting a prejudice, which it will require all his art to remove. We cannot help drawing consequences from the very position which the lines accidentally happen to take in that particular figure which we reason upon, though this particular position make no part of the supposition. The student may convince himself of this, if he read the fourth proposition by the assistance of the figure which belongs to it; and then again making use of the two triangles ABG and ACF in the figure to proposition fifth; making the same supposition in both cases. If the consequences are suggested to his mind more readily by one set of figures than another; this can arise from nothing but the stress laid upon the magnitude and situation of the triangles, which, by the very supposition have no particular magnitude or situation. But this prejudice will be lessened, and gradually removed by varying the position of the figures, turning them upside down; and by such other methods as will readily occur to the attentive reader; but above all by enlarging the figures, setting the imagination to work until they cease to be the objects of our senses. The center of the moon, when
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in the equinoctial, joined by straight lines to the two poles of the Earth, would make a very proper isosceles triangle with which the student might finish his speculations upon this proposition.

C H A P. VII.

Concerning indirect demonstrations.

INDIRECT demonstrations are generally used, when the proposition is the converse of some other which has been already demonstrated. A proposition is said to be the converse of another, when the hypothesis or supposition in the one is the thing to be demonstrated in the other, and the contrary; thus the converse of the fourth would be; supposing the base BC equal to the base EF; and the angle ABC equal to the angle DFE; as also, the angle ACB equal to the angle DFE; that then the two sides BA and AC will be equal to the two sides ED and DF, each to each; viz. those sides which are extended under the equal angles, that is AB equal to DE and AC to DF; as also the angle BAC contained by the two sides equal to EDF.

Now this proposition admits of a direct demonstration; for if the point B be put upon E and BC upon EF, the point C will apply itself to F, because BC is equal to EF; and BC applying itself to EF; also BA will apply itself to ED, because the angle ABC is equal to DEF. Certainly for the same reason CA will apply itself to FD, that is, because the angle ACB is equal to DFE: the point A will therefore apply itself to D; and so AB will be equal to DE and AC to DF and the angle BAC will be equal to EDF. &c.

But if we suppose the angles at B and C, not only equal to those at E and F, but also equal to one another; we may prove AB to be equal to DE as before; and then by putting B upon F and BC upon FE we can prove that C will apply itself to E; and that the line AB will be equal to DF: but it has been demonstrated that AB is equal to DE, therefore by the first common notion DE is equal to DF; and no doubt some authors would produce this as
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a demonstration of the sixth proposition ; but it is liable to the same objections, as it converse, which have been mentioned already.

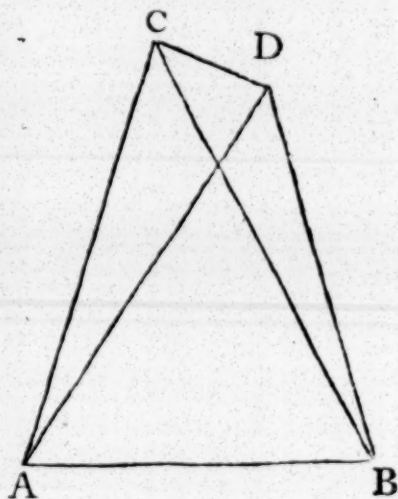
But instead of going any such way as this to work Euclid has given a very elegant demonstration of the sixth proposition, which is the converse of the fifth ; which the reader may please to turn to and examine ; after which he may proceed with the following remarks.

And first it will be worth his while to observe the ingenuity which Euclid has displayed, in making the angles which are equal by the supposition, the angles between the equal sides of his two triangles ; and BC being a side common to both it remains only for him to make BD equal to AC ; and to join DC. And here it may be asked, and it is also the most important question which can be put with regard to this proposition, how comes it to pass that he cannot find by his construction that the points D and A are the same ? It is true such an instrument as the compasses would find out this if it discovered any thing ; but this use of the instrument is rejected for reasons given already. For the solution of this difficulty therefore the student may please to turn to the third proposition, and consider the apparatus made use of, for cutting off, a part from a line equal to another : and he will find that the supposing the problem possible implies that the last described circle must cut AB that is that the point D must be between A and B. And when we find that this supposition leads to an absurdity what are we then to conclude, but that the problem is really impossible ; which never can be if AB is greater than AC ; therefore it is not greater. And I appeal to any one, whether the demonstration is either difficult, obscure or inconclusive when it is considered in this manner. Nothing can be so senseless as the objections usually made to indirect demonstrations. Every demonstration may be loaded with objections until it becomes indirect : the reader will find a specimen of that kind in my remarks upon the first proposition ; where a caviller is introduced denying that the equilateral triangle there described is a fixed magnitude. But our author never carries his reasoning farther than good sense requires ; so that it is only such propositions as this sixth, which take an indirect form in his hands. And this kind of reasoning will not apply unless we know
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how many ways the thing may be ; as in the present instance AB must be equal to AC or longer or shorter : or else we must get at such direct consequences from one part of the supposition, as will overturn the other : as in the next proposition, which I must now beg the reader to peruse : after which I shall be ready to lend him my assistance in removing such difficulties as usually lie in the way of beginners.

It is supposed here, that AC is equal to AD ; and at the same time BC equal to BD. The learner will do well to fix his attention entirely upon this : first tracing the consequences which follow from the supposition of AC's being equal to AD as far as they will go ; which he may easily do ; as it is only that the angle ACD is equal to ADC.

He is next to observe the consequences which follow from the position of the lines ; and these consequences are, that the angle ACD is greater than BCD, which follows from this common notion that the whole is greater than its part ; and also that the angle BDC is greater than the angle ADC, which follows from the same principle : and with a distinct impression of these things upon his mind, it is impossible to miss the conclusion that the angle BCD is much less than BDC : and this point being once gained ; he is next to turn his thoughts to the other part of the supposition ; in which it is pretended that BC is, also at the same time, equal to BD : and the consequence which follows from this part of the supposition cannot fail to engage his attention ; being no other than this, the angle BCD is therefore, by the fifth proposition, equal to BDC : When we suppose AC equal to AD we must conclude that the angle BCD is much less than BDC ; but these very angles must be equal upon the supposition that BC is equal to BD ; is not this saying in the strongest terms that the two suppositions are inconsistent with one another ; that is, that it is impossible for AC to be equal to AD ; and at the same time, BC to be equal to BD. Which was to be demonstrated.



It

It will be a very necessary and useful task for the reader to set himself after he has finished every demonstration ; to examine whether some consequence has been drawn from every part of the supposition ; for though it may not always be convenient for the author to draw them, yet the reader should always do it for himself ; for if the suppositions have no consequences to the purpose either expressed or understood ; they ought by all means to be omitted ; and if the author trusted to the reader's ingenuity for *finding them out* he neglects his duty if this part of his business be overlooked. The necessity of this practice being thus made evident ; I shall explain my meaning by a particular example ; it is said that AC and AD are to have the same extremity ; but no use is made of this expressly in the demonstration : and yet without this the demonstration could not proceed ; because ACD could not be a triangle unless these lines had the same extremity : neither could it be a triangle unless C and D were different points. It is moreover said that they are to be towards the same parts : now if the points C and D were on different sides of the line AB ; the angles ACD and BCD would either be two distinct angles and the one not a part of the other ; or if BCD be a part of ACD, then ADC would not be a part of BDC ; which is absolutely necessary for bringing out the conclusion which we aim at. Lastly it is said that the equal lines are to be terminated in the same extremity ; and without this neither ACD nor BCD would be isosceles triangles ; and then no inconsistency could follow, reason as long as we pleased. All this will be obvious by taking the points C and D on different sides of AB, and joining CA, CB ; DA, DB and CD ; so that CD may cut AB or fall beyond the point B. And indeed it is very difficult to understand any general proposition, without some representation of all the different positions which the lines can take : and the reader cannot finish this proposition better than by supposing the point D within the triangle ACB, and to assist his imagination, by making AC and AD equal with a pair of compasses ; his demonstration will rest upon the same principles ; only producing AC and AD ; he is to reason upon the angles below the base ; but every thing else will be the same as in Euclid's demonstration.

I shall conclude this chapter with my opinion of indirect demonstrations, which some very ignorantly object against. It would be absurd to speak of the certainty of a demonstration: as to their use therefore, considered as an exercise to the mind, I think they are preferable, for several reasons to direct demonstrations. Because in a direct demonstration, there is generally some circumstance which catches the senses; and not being too difficult of persuasion, where our own ease is so much concerned, we are disposed to rest very well satisfied with that, though it may convey, but a very imperfect notion of the full extent of the demonstration. But the contrary happens in such demonstrations as are indirect; for the reasoning being generally at variance with the sensible representations of the magnitudes, it requires some effort of the mind to get beyond the prejudice of sense, so as to comprehend the force of the reasoning: and when we cannot conquer our indolence or command our attention, it is a very decent excuse to lay the blame upon the nature of the demonstration. But I have been so particular in my examination of these two propositions, that the attentive reader can hardly want any farther information upon this head.

C H A P. VIII.

Of geometrical demonstration.

IT will now be proper for the reader to endeavour, from a careful examination of these first seven propositions, to collect as accurate a notion as possible of a demonstration, according to the rules which Euclid has prescribed to himself.

We are apt to satisfy ourselves of the truth of things, in the easiest and shortest way we can; and even when we chuse to consider the properties of magnitude, instead of having recourse to some general principles, we trust to mechanical instruments; and very often to vague conceptions collected at random from accidental observation. But our author's plan is very different from this: The definitions, postulates and common notions, are the only foundation upon which his geometrical reasonings are founded;
and

and when he mentions any thing as a property of a figure, he considers it as incumbent upon him to shew that it follows from or agrees with the definition of that figure ; and this he does by the assistance of some common notion, postulate or other definition. And whatever property is not reducible in this manner, he considers as impossible to be demonstrated ; though it might be a very obvious truth from other principles. Truth therefore is not so much his object as consistency ; it does not appear directly to be any part of his business to collect as many useful facts as possible concerning the properties of figures ; but only to be convinced that what he does produce have a solid foundation, making every part acquire additional strength by the consistency of the whole.

He will never allow it to be said that such a property is so simple as not to require a demonstration ; for to alledge this as a reason for taking any thing for granted, would be, upon his plan the greatest absurdity. For it may be said, if it be so very plain and obvious, whence does it derive this obviousness ? If it be the property of a figure ; that figure is defined to be so and so. How does it appear that it is consistent with that definition ? If you point out the consistency, you have demonstrated the proposition ; but if that cannot be done, it may be true according to your principles, but mine lead me absolutely to reject it.

That this is Euclid's plan will appear from the whole of his work, even to the astonishment of the attentive and judicious reader ; when it is considered how his invention must have been perplexed, by the obvious, but inaccurate experimental tests, which would for ever be presenting themselves to his imagination ; and how his patience would be tired out by the number of steps, which he foresaw to be necessary, before he could arrive regularly, at some conclusions seemingly very simple indeed. A man would feel himself in a very awkward situation, who should set about measuring magnitudes, without being able to judge when one straight line was longer than another ; and yet it is wonderful, by how many steps, this very test is removed from the first principles ; and which are all necessary for giving it a scientific stability. We find this delivered in the nineteenth proposition of the first book ; but as this is rather foreign to my present purpose, and beyond the sup-

posed knowledge of the reader, it will be sufficient to have mentioned it by the bye, only as a fact to illustrate my meaning. I shall therefore now beg the attention of the student, while I review the first seven propositions, in support of my opinion concerning Euclid's plan of demonstration.

In the first proposition his principles are the third postulate; the first postulate; the fifteenth definition; the first common notion; and lastly the twenty fourth definition. He is not satisfied to make an equilateral triangle any how; but he makes it by the assistance of his own instruments and principles. The second proposition is constructed and demonstrated by the first postulate, the first proposition, the third postulate, the fifteenth definition and the third common notion; and we have also here a proof how tenacious our author is of his principles; for it is not absolutely necessary that in this second proposition, an equilateral triangle should be described upon AB; because an isosceles triangle would have answered the purpose; for no consequence is drawn from AB's being equal to AD; as it would be sufficient if AD and DB were equal; but he could not make an isosceles triangle, consistent with his own plan and principles. The third proposition rests upon the same foundation; its demonstration being supported by the second proposition, and the third postulate.

In the fourth proposition the principles which we reason from, besides the supposition, are the eighth and twelfth common notions; to which every part of the demonstration may be reduced; for the application of the triangles to each other must be allowed to be a proper principle to reason from, otherwise it is difficult to conceive what use can be made of this eighth common notion, which is the definition of equality; it will not matter much whether this putting of the triangles be reckoned a construction or not; I am rather inclinable to consider it as a thing of its own kind, and not to be understood either as an hypothesis or construction: I know Euclid makes use of the expression the point being put, which seems to imply a kind of construction; and yet it would be doing no great violence to the language, to paraphrase it in this manner; the triangles have no particular position; that is they have no relation to any points, lines or angles, which can be affected

affected by our giving them a particular position ; let us suppose them then to have such a position, as it is necessary magnitudes should have before that eighth common notion can be of any use ; that is let us suppose their situation such, that the one is applied to the other ; and the point A put upon D &c.

It is plain that Euclid himself found it impracticable to give this principle a more scientific form ; and therefore instead of making a definition of it, left it among the common notions ; but whatever opinion we adopt ; it will answer our purpose, because every thing is inferred from the principles, which are laid down as such. If we were required to place triangles upon each other, which had a determinate position before, I would certainly consider it as a construction ; but in the present case, I consider it only as a necessary apparatus before any consequence can be drawn from the eighth common notion. And as to the objections, upon a conceit that this is a mechanical application, they have been very fully answered already ; so that upon the whole there is nothing in this proposition, but what agrees with the notion of a demonstration which has been delivered in the beginning of this chapter.

The fifth proposition seems by the references to depend upon the four first propositions and the third common notion ; but as the construction required is only the simplest case of the third, the first and second propositions are not necessary for cutting off AG equal to AF. But the sixth requires the four first propositions ; because the simplest case of the third, will not be sufficient for cutting off BD equal to AC ; and, what I must beg the reader particularly to attend to, neither can it be done by a ruler and compasses ; because there is something singular in this construction : for it is here required to do, what we afterwards find impossible to be done ; and the unlucky circumstance is, that whoever trusts to a ruler and compasses will come to the knowledge of this too soon ; a pair of compasses lets him into the secret immediately, before his mind is prepared for it by any scientific information ; and hence will arise a gross misconception of the author's meaning ; because the reasoning must appear useless and unnecessary, being intended to discover to a man what he has found out already. In all such cases as this therefore our ruler and compasses are to be laid aside,
and

and some method devised which may be more consistent with the nature of the postulates as here applied : for by keeping strictly to the information which we get from them, we never can discover that we have been attempting to perform what is impossible, before we are led to the absurdity of concluding the triangle ABC to be equal to DCB. I would recommend the performance of such constructions as this, by the hand unguided by any instrument ; as this will favour the condition of the proposition, because such a construction can make no discovery before the proper time. Not that I would *suppose* it performed ; as that might foster prejudices, but rather represent every step of the construction ; BC being joined already I would describe upon it an equilateral triangle producing the sides &c. as directed in the remarks on the third proposition, until the last described circle cut AB in some such point as D ; because the whole construction will be necessary to convince the reader that he has no scientific notice of the impossibility of the problem from the construction, but must wait for it until he come to the absurdity. And hence it appears that Euclid requires every part of this proposition to be referred to the first principles ; contrary to the sentiments of those superficial readers, who imagine that they have only to suppose a point D, and the thing is done.

The demonstration of the seventh is reducible to the fifth ; and the ninth common notion. And thus it appears, that Euclid's intention is not to shew that the proposition is true in any manner ; but that it is immediately connected with, and depends upon his principles, or in other words that his aim is not to persuade but to demonstrate.

C H A P. IX.

In which is explained the geometrical meaning of the words finite and infinite.

“ THE human understanding, *says Bacon*, shoots itself out,
 “ and cannot rest ; but still goes on though to no purpose. Thus
 “ 'tis inconceivable there should be any bounds to the universe ;
 “ yet it constantly, and, as it were, necessarily recurs, that there
 “ must

“ must be something farther. So again it cannot be conceived how
 “ eternity should have flowed to the present time: and there is the
 “ like subtilty as to the infinite divisibility of lines, &c. all arising
 “ from the weakness of human thought.” And whoever chuses to
 indulge this *weakness of thought* will be put into a fine train by
 reading what *Locke* has advanced upon the idea of infinity. But it
 is the end of all mathematical learning, instead of lending any
 assistance to encourage this kind of dreaming, to root out such a
 disposition from the human mind; and instead of such fooleries,
 to give it something to employ itself upon suitable to the powers
 and faculties of a human soul. An understanding which has any
 force will always shake off such dreams; the indulging in which
 is the sign of a weak intellect. For the philosopher and the idiot
 agree perfectly in their notions of infinity; and the difference be-
 tween them is only discovered when they come to a comparison of
 determinate things and quantities.

The geometrical notion of infinity, as far as it regards extension,
 is derived from the second postulate; let it be taken for granted
 that a straight line may be continued directly forward: The geo-
 metricians never trouble themselves with multiplying any assign-
 able parts of extension, in order to be convinced that after they
 have advanced millions of miles, they are still as far from the end
 of an infinite line as they were at their first setting out.

The two straight lines DE, DF in the second proposition, are
 infinite in the full and proper geometrical sense; and so likewise
 are AD and AE in the fifth; and the third proposition may be
 applied to construct this, because AF is less than AE: but why is
 it less? Because AF is a finite and AE an infinite line: and this
 instance fully explains what is meant by an infinite line in this
 science; for it means only a line longer than any determinate line
 which they have occasion to take; and farther than this they give
 themselves no concern. In the ninth proposition AB and AC are
 infinite lines; and the point D is taken that we may have a fixt
 line AD and so in other instances.

I am now arrived at the conclusion of this dissertation, in which
 I have endeavoured to point out such circumstances, as the thought-
 less reader is apt to overlook. If it should seem strange to any one to
 find

find such a cerimonious introduction to a simple, well defined and demonstrative science, he will be pleased to observe some peculiarities which distinguish this subject remarkably from most others. It is true the language is plain and significant; but the truths to be communicated are conveyed in too few words to engage the attention of those who have been accustomed to the figurative language in which most other subjects are delivered; and where the reader, instead of having consequences deduced from every word, carries something with him if he attend to but one word of three. And for this reason I have been a little diffuse and circumlocutory; that the transition, from the common form of speaking, to this very concise and accurate method of expression, might be made with greater ease by the reader, after some part of the subject has been explained to him in his own way. But I must beg leave to introduce the student to Euclid himself, and defer all future intercourse with him until he has reached the end of the first book.

DISSERTATION III.

THE first book of Euclid's elements will necessarily suggest a great variety of reflexions to a judicious reader; he will perceive a particular method of arrangement, and a particular manner of demonstration which makes that arrangement absolutely necessary; with something characteristic even in his way of constructing problems. But an indolent reader requires to be put in mind of each of these particulars, otherwise he will be disposed to overlook them. Some remarks therefore upon each of these heads may be useful to put him into a proper train of thinking; and this shall be the subject of the present dissertation.

CHAP. I.

In which Euclid's method of demonstration is proved to be necessary contrary to the opinion of Clairaut.

“ Qu'Euclide se donne la peine de démontrer, que deux cercles
“ qui se coupent n' ont pas le même centre, qu'un triangle ren-
“ fermé dans un autre, a la somme de ses côtés plus petite que
“ celle des côtés du triangle dans lequel il est renfermé; on n'en
“ fera pas surpris. Ce Géomètre avoit à convaincre des Sophistes
“ obstinés, qui se faisoient gloire de se refuser aux vérités les plus
“ évidentes: il falloit donc qu'alors la Géométrie eut, comme la
“ Logique, le secours des raisonnemens en forme, pour fermer la
“ bouche à la chicane. Mais les choses ont changé de face. Tout
“ raisonnement qui tombe sur ce que le bon sens seul décide d'
“ avance, est aujourd'hui en pure perte, & n'est propre qu'à ob-
“ scurcir la vérité, & à dégoûter les Lecteurs.”

This is a very singular opinion concerning the motives which led Euclid to that rigorous method of demonstration which he has adopted : for we are here told that it was not choice, but the circumstances of the times in which he lived, which brought him to write as he has done : his living among sophists, drove him beyond the bounds of good sense : for if he had been left to follow his own inclinations, we have here room to suppose that his demonstrations would have appeared in a very different form, or in other words, that had he been a Frenchman and lived in the same polite and happy times as *Clairaut* he would have written just as he has done. The French would have taken his word for it, that two circles, which cut one another, could not have the same center; and that polite nation surely never would have contradicted him if he had said the same thing of two Ellipses : but one thing I am certain of, that, had they been acquainted with no other principles but such as this author would have us appeal to, they never could have contradicted such an assertion upon any good grounds.

Now what I mean to prove, in opposition to this doctrine, is, that Euclid is remarkable for adhering to the very principles which *Clairaut* thinks he himself has gone upon : and I shall now produce as many instances as I think the reader's patience can well endure in proof of this assertion, namely, that Euclid every where distinguishes himself by keeping clear of such things, as good sense would decide of itself before hand. Now as the method of these two authors is not simply different, but directly opposite the one to the other ; *Clairaut* and I must assign a very different office or employment to this same good sense, the presuming to teach which, he says, answers no other end but to disgust the readers. But I am certain good sense will never be either affronted or disgusted to have any thing set in a better or stronger light than what it appeared in before ; and I challenge any one to produce an instance, where the thought is not improved and rendered more accurate by Euclid's reasoning ; and it must be a strange kind of good sense, which could reckon such reasoning, as he says *en pure perte*.

But as to his looking upon himself as bound to convince obstinate sophists, it is very obvious that he never went in the least out of his way on their account ; and guided himself by very different
maxims,

maxims, as will appear to every one who considers his manner of reasoning, which is to the last degree inconsistent with any such supposition, as that he had the least intention to combat such prejudices as the sophists raise, when they have a mind to sport with the credulity and ignorance of the multitude. It is true he furnishes us with the means of escaping out of their hands, if a proper use be made of his principles; but it never once entered his thoughts to combat their silly objections, which never could be obtruded upon any one, who is not entirely ignorant of the nature of the subject.

However, as a full proof of what I now assert, I would recommend it to those who may chuse to differ from me in opinion, to read *Proclus's* commentary, who sets himself in earnest about this hopeful task; and there, it is granted he will find reasonings, *qui tombe sur ce que le bon sens seul décide d'avance*; and I will allow that such reasonings are of no other use but to obscure the truth, and disgust the readers. The dispute therefore is brought to this short issue; let the reader peruse the first book of Euclid and then read *Proclus's* commentary, observing the difference between the two kinds of composition; and I am certain he will acquit Euclid of any design to convince obstinate sophists; and moreover must be persuaded that he has not only omitted every thing which good sense can *intuitively* determine; but that he has also left several things for the reader to supply which would not come easily under his own precise notion of demonstration; for Horace's rule was never better nor more properly applied than by this author;

— et quæ

Desperat tractata nitescere posse, relinquit.

Several instances of which, taken from his first book, I shall now proceed to enumerate, not considering them as oversights, but as proofs of a most refined and accurate judgement.

And first, to begin with that famous principle by which the meeting of two straight lines is determined: We may readily suppose that Euclid introduced the seventeenth proposition, with a view either to demonstrate or explain the eleventh common notion,

as being the converse of it. For from this proposition we learn that whenever the two lines meet, the angles made by the cutting line, upon that side, are less than two right angles; but we can by no means infer from this that the lines will meet whenever the angles are less than two right angles. But again he has demonstrated in the twenty eighth proposition; that when these angles are equal to two right angles the lines will never meet. But he cannot, from this say, that this is the only case in which the lines will never meet: only thus far he may go, that the supposition, from which he infers that they will never meet, requires absolutely that the angles should be exactly equal to two right angles; and the least deviation from this supposition will render his principles of no avail; as no consequence can follow from them; for their absolute equality to two right angles and nothing less, is necessary to prove that the lines will never meet. Now this is sufficient at least to ground a persuasion, that in all other cases they will meet; especially when it is considered that such lines are inclined towards one another, as may be shewn by drawing a perpendicular to one of the lines; which will prove that they are approaching towards one another upon the side where the angles are less than two right angles: and going farther from one another upon the side where the angles are greater than two right angles; but the opinion thus acquired is not of the nature of a demonstration. Nor could he even say that a line intersecting one of two parallels, would meet the other; and indeed if this could be shewn upon scientific principles, it would be a demonstration of the thing in question: but this would be taking the thirtieth proposition for granted, the truth of which depends upon this very principle.

We may therefore suppose that such things as these and numberless others of the same kind must have occurred to a mind so fruitful in expedients upon such occasions, and were all rejected by him as falling infinitely short of his idea of demonstration: wherefore without perplexing his reader with impotent attempts towards a demonstration; he judged it more proper to cut short this fruitless search by placing this principle among the common notions; and as it appears from this investigation that it was not placed here without reason; so it has a right to keep its place, until it can be
shewn

shewn that some absurdity can follow from the use of it. And upon this point I have only to add, that it is certain from the general tenour of his demonstrations, that if he could have exhibited it without any scientific defect, but in the clumsy dress in which some authors have arrayed it, he would have rejected the conclusion with disdain, and left it where we now find it. So that it is not a demonstration of any kind which was his object, but such an one as is both elegant and forcible.

But further, I have mentioned before, what seems to be an exceeding good general rule of criticism in this science, that we ought always to examine if some consequence has been drawn from every part of the supposition or construction. Now there are several instances where Euclid has laid no stress, even, upon the most material part of the supposition; which shews how little he minded the conviction of those *obstinate sophists*, for whose sake according to *Clairaut* he gave his elements such a formal dress, to the great disgust of every Frenchman; because here would have been so good a pretence for standing still, that no arguments could have persuaded these sophists to proceed while such blemishes remained. But to come to particulars.

In the twenty second proposition, it is required to make a triangle of three given straight lines, but with this necessary limitation; that any two of them must be greater than the third. However no use is made of this limitation in the demonstration of the proposition; and I am persuaded that our author omitted to prove that the two circles will cut one another, which depends upon this limitation, because this proof does not admit of such a natural and elegant turn, as he had determined to give to every step of his demonstrations; there being no direct principle to refer to, by which the intersection of circles can be proved; that they have some space in common is the principle from which their intersection is to be inferred, which is indeed a common notion but not one of those, which he has selected to draw consequences from; concluding I suppose that all rational men, could make as ready and extensive an use of such notions as these, as he could, and therefore, that it would be unfair in him, to appropriate to himself either the notions or their consequences; or according to *Clairaut*,
that

that such reasonings would only fall upon, what good sense had decided before hand, in a manner full as satisfactory.

Again, there is an instance of the same kind, in the twenty fourth proposition; where the most material part of the supposition is never once mentioned in the demonstration; viz. that the angle BAC is greater than the angle EDF; consequences are tacitly drawn from it, like the taking for granted that the circles will cut one another in the twenty second; for without this part of the supposition the angle EGF would not be a part of DGF, nor, if the point F was within the triangle DEG would this angle EGF be a part of the angle below the base of the isosceles triangle DFG; which is absolutely necessary in order to prove that the angle EFG is greater than EGF; or, in other words, it follows from this part of the supposition, that DG always falls upon the same side of DF which it is represented to be upon in the figure; but it would be difficult to give this *reason* such an elegant turn as to entitle it to a place in one of Euclid's demonstrations.

But the sophists would stick at much less matters than these; they would require him to prove in the first proposition, that the two circles would cut one another; and that their intersection is a point: and in the second that there would always be such points as G and L: also in the fifth that AF and AE were two such lines, that a part could be cut off from the one equal to the other, or that AE is greater than AF: they would likewise find several parts of the supposition in the seventh from which no consequences are drawn, which I have taken notice of before; in the twelfth proposition they would require him, to prove that a circle described with C for the center, and through any point on the other side of AB, would cut the line AB; which would give occasion to his making use of the infinity of the line from which no consequence is drawn, as the demonstration now stands: They might likewise ask him, upon the corollary to the fifteenth proposition, whether when two lines meet at a point they make angles at their meeting equal to four right angles: and in the thirtieth proposition, by what principle he makes a straight line pass through three points, that is, takes it for granted that it may pass through three points, viz. a point in each line; for he ought to have taken two, and to have

have proved by the eleventh common notion that there would always be a third: Nor would it be very impertinent of them to require him to prove that AH and FG will meet in the forty fourth proposition; and indeed in that instance I cannot help being of opinion that the construction would have been more in Euclid's manner if he had made GH equal to BA and then joining HA had proved that HA was parallel to GB by the thirty third proposition. In short if it had been his intention to write for the conviction of the sophists, his demonstrations would have had a very different form from that in which we now find them. And it will be easy for any one who has read even the first book with attention to add a great many more to the above catalogue of omissions.

But I hope nobody will so far mistake my meaning here as to suppose that I consider such questions as these, either as captious or improper. They are mentioned only to shew that Euclid had some plan by which he directed himself in his demonstrations, very different from the supposition that it was to convince obstinate sophists. For I am so far from considering such questions as improper; that I think they are such as every one must ask himself, and a great many to the same purpose, otherwise I am certain he will have a very imperfect notion of the book; and whoever is in earnest to understand this subject should take every opportunity of devising such objections; the proper answers to which will be readily suggested by what Euclid has said. But this should be set about upon some regular plan; and I would therefore recommend it to the student, after he has finished the first book to give it a second reading, examining every supposition and construction, and the consequences drawn from them; and their full importance in the proposition; which is not to be determined from the number of words in which they are expressed, but by the dependence which the conclusion has upon them: for I have an instance just now in my view, in which I may safely say that the first reading hardly ever discovers the full importance of the supposition: what I mean is the supposition in the forty seventh proposition; from which the consequence that follows is; CA and AG as also BA and AH make but one straight line. Now whoever has read this proposition without perceiving that the conclusion depends entirely upon

upon this, cannot be said to have understood the demonstration; and that this is true appears; because the square BG is only double of the triangle BFC because GAC is a straight line. The reader is also carefully to observe whether the conclusions are general or particular; because it may so happen that a particular position of lines or points will lead to a conclusion, which will by no means follow if these are changed; and often if it do hold good it may require some new steps to come at the conclusion. But it seems necessary to illustrate this by particular examples.

In the eleventh proposition of this first book, it is required to draw a perpendicular to a line, from a point given in it. This may be divided into two cases; for the point may be at some distance from the extremity, or it may be the extremity of the line. Euclid has considered the first case only; and there is but one false consequence which could be drawn, by taking the other for granted; and what is very remarkable *Simson* in his edition has hit upon this very consequence, in attempting to prove that two straight lines cannot have a common segment; because it must be taken for granted that two straight lines cannot have a common segment before a perpendicular can be drawn from the extremity of the line; for as the line must be produced, without this limitation, it may take two directions.

Again in the twenty fourth proposition, the point F may have three different positions; it may be without the triangle DEG or in the line EG or lastly within the triangle: For a particular instance the student may make such a triangle as ABC in the figure to this proposition; having AB longer than AC; and after the triangle DEG is made; with the center D and distance DG describe a circle; which will cut EG; now any straight line drawn to the circumference of this circle, from the point D, within the angle EDG, may represent the line DF; from which the three different positions will be obvious. When the point F falls within the triangle, the conclusion may be inferred from the twenty first proposition; but it would be more uniform to produce DF and DG; and to reason upon the angles below the base, in the same manner as Euclid has done upon those that are above it. It is true *Simson* reduces this to one case, but I think not in the manner of Euclid;

for

for there is no circumstance given by which we can determine which side is the longest; there ought therefore to be added to the present supposition, *AC being longer than AB*; but the determination of this point I shall leave to the curious. But farther I would even recommend the same kind of examination, where there was no probability of finding any variation; as for instance in the sixteenth proposition, the conclusion depends upon FCD's being an angle; for if that could ever cease to be an angle, the outward angle might be equal to the inward opposite one; but that is impossible because then the point F would be in BC produced; and consequently the point E upon C that is the middle of a line at its extremity, which is absurd.

Having been thus particular in shewing what is not Euclid's method of demonstrating, I shall conclude this chapter with a few instances tending more directly to explain what it is.

And his great peculiarity seems to be a determined resolution always to refer directly to some principle, and never trust to a vague conception; or more properly never to make use of a vague expression: The thirteenth proposition is a remarkable instance to this purpose; That the angles are equal to two right angles strikes the senses immediately and produces a conviction which has something fluctuating in the nature of it; arising from observing that the angle ABD is above a right angle, by just as much as ABC is less than one; now this is not a principle sufficiently distinct to reason from in a demonstrative science; and Euclid has shewn great art in the demonstration of this very proposition; which he has reduced to the common notion, *magnitudes which are equal to the same are equal to one another*; and how accurately he keeps up, through the whole demonstration, to the notion of the angles being magnitudes, cannot fail to engage the attention of a judicious reader. The twentieth proposition furnishes an instance to the same purpose: every one believes that two sides of a triangle are greater than the third; and he may take up this opinion from a consideration of many different circumstances; he may consider that a straight line must surely be the shortest way between two points: or he may trust to the judgement of the aís of the *Epicureans*; when a bundle of hay is placed at one of the angles; but

still he will find something unsettled in his conviction as resting upon no determinate principles; and Euclid never leaves any thing which he takes in hand to demonstrate, in this unsettled state: he makes this property follow, not from any unstable principles or random conceptions, but from the very nature of a triangle; and a thorough examination, of the whole apparatus necessary for the demonstration of this principle, will give the intelligent reader a wonderful insight into Euclid's method of demonstration.

It seems almost a desperate undertaking, to endeavour at reviving a taste for accurate demonstration; because the common vague conceptions of magnitude, proped by a ruler and compasses, when they begin to totter, are judged capable of performing every thing that can be expected from this science. But if a reformation is at all practicable, it seems most likely to be brought about, by turning the attention of the world to Euclid's method of demonstration; and I ground my hopes of success upon two circumstances; namely that the modern writers of elements of Geometry neither understand the nature of their own demonstrations, nor the force of those of Euclid. Because if they understood the force of Euclid's they would be ashamed to dignify their own with such a name; and if they understood the nature of their own, they must perceive that a great deal of expence both of time and thought might be saved by delivering their propositions as facts, which had been demonstrated; unless indeed we suppose that what they call demonstrations are added merely in compliance to vulgar prejudice. Now could a judicious reader, no ways interested in the issue of the debate, be once brought to weigh the merits of such demonstrations; he must either decide in favour of such as Euclid's, or declare that all demonstration was now become unnecessary; and that therefore the farce of it should be laid aside. But upon such an event, such is the charms of truth, I would naturally expect a considerable number to leave their new masters, as soon as they had relinquished their pretensions to demonstration: and it is upon such hopes only that I proceed to give some other instances as characteristical of Euclid's method of demonstration.

Euclid's demonstration of the seventh proposition, may be compared with the one given in *Deschales's* edition of Euclid; which

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was no doubt intended as an improvement in some respect or other. And this seems to me the more extraordinary because the demonstration of the seventh is remarkably distinct and pertinent; leading to the conclusion so directly that the full force of the demonstration must be perceived by the mind: and I am persuaded that our author would have given up all thoughts of writing his elements, if he had found himself obliged to rest such a material part of his science upon so unstable a foundation, as the description of two circles, with an indeterminate radius and even before it could be known in how many points their circumferences would cut each other; because every point of intersection, that did not lie on the other side of AB would contradict the proposition; and prove that two triangles might be placed according to the supposition, until these points were reduced to one, which is not so easy to do as a superficial reader may imagine. But there is nothing which I have observed Euclid more cautiously to shun than the describing a circle with an indeterminate radius; or more properly, than the fixing a line, otherwise undetermined, for the purpose of describing a circle. It is true he makes use of consequences which must be derived from circles described in this manner; but this is supposed to be transacted in such a manner as to bring no disgrace upon the science. In the sixteenth proposition AC the side of any triangle is required to be cut in halves, which cannot be done until the line is fixed; but then I immediately perceive that, notwithstanding this, as no consequence is deduced from AC's having any particular length, the demonstration is nevertheless general; so that I can finish my construction and have AC as undetermined as I found it; so that this transaction is supposed to be carried on in private by the reader himself; and which he acquiesces in, as bringing no reflexion upon the science.

There is another thing which I shall just observe before finishing this chapter: Euclid's demonstrations are often more general than they seem to be; for whenever, it would be only a repetition of the same steps, he always omits them, whether in a construction or demonstration. The forty fifth proposition furnishes a remarkable instance of this; he shews how to turn a four-sided figure into a parallelogram &c. and as no new circumstance

would occur, whatever be the number of sides of the figure; he concludes that it is general, without adding a word more; which has misled some ignorant conceited people to think that they improve upon Euclid by shewing a simpler way of turning a four-sided figure into a parallelogram; never considering that Euclid has turned any rectilineal figure into a parallelogram, whatever be the number of its sides; and whatever be their position to one another. Now I hope it is evident that Euclid had some other plan in his demonstrations than to convince the obstinate sophists; and what that plan is I shall explain in the next chapter.

CHAP. II.

Of the arrangement of Euclid's propositions.

THOSE who know nothing of geometry tell us the science should be delivered according to the following arrangement. We should first begin with *a point*; and after having laid before the reader all its various properties, such as position, want of dimensions and the like, we are next to proceed to the straight line; and after delivering every thing that can be said of a single straight line, we may then advance a step farther and take two of them under our consideration; first enquiring into the properties of such as are parallel; and this subject being exhausted we may next examine two straight lines that meet so as to form an angle; which will furnish us with a fruitful source of speculation; because this will now be the proper place to treat of angles; to shew how they may be proved to be equal, and into how many kinds they ought to be divided, not forgetting to prove that a right angle consists of ninety degrees. Now supposing the properties of straight lines exhausted, the natural transition is to such as are crooked, arranging them into different orders, beginning with the circle; but take care not to consider it as a figure; for that would destroy all order; for it is the circumference only that belongs to our present subject: we may next proceed to the conic sections, and so on to other lines.

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The business of lines being finished, there would be abundant matter for dispute, when we came to treat of figures, whether we should begin with the triangle or the circle; with the triangle as being the simplest rectilinear figure; or with the circle as being bounded by one single line; the circle might probably have the preference upon a full examination as being a *perfect figure*, and for other reasons which it would be too tedious to insist upon at present. But supposing this difficulty got over; the triangle would certainly claim the next place; and of all the different kinds of triangles, the equilateral would put in for the first; the isosceles would deserve to be considered next; from the isosceles the transition must be to the scalene; taking care all the time, in delivering their properties to say nothing of their angles, for that would be to confound the nature of things; because they are afterwards to be divided according to their angles, right, acute and obtuse; and this new division would furnish the proper place for that part of the subject. From triangles the most natural progress would be to the square, and then to figures less regular. And this is the true philosophical arrangement.

Such are the engines which *Folly* has been erecting for battering this noble work of our author; seconded with other auxiliary schemes, the force of which seems leveled against it, though more indirectly. But what holds the greatest vogue at present, as falling in with our indolent disposition, and bids fair for driving him to the last extremity, is the proposal of considering no more of the subject than what is absolutely necessary; with the inviting title; *as applied to such and such useful purposes of life*. Elements of geometry carefully weeded of every proposition tending to demonstrate another; all lying so handy, that you may pick and chuse without ceremony. *This is useful in fortification*: you cannot play at billiards without this. You only look through a *telescope* like a *Hottentot* until this proposition is read; with many such powerful strokes of Rhetoric to the same purpose. And upon such terms, and with such inducements who would not be a *mathematician*? Who would go to work with all that *apparatus* which I have described as necessary for understanding Euclid; when he has only to take a pleasing walk with *Clairaut* upon the flowery banks of some delightful river.

river, and there see with his own eyes, that he must learn to draw a perpendicular, before he can tell how broad it is: such is the ingenuity of these Frenchmen,

Ut puerorum ætas improvida ludificetur,
Labrorum tenuis.

I had formerly the ambition to join myself to this fraternity, and taking a walk in the *french stile* along the banks of a river, proposed to teach the reader, how to measure its breadth by his hat; but I soon involved him so much in the properties of triangles, that I despaired of making my point good, in the true elegant and undemonstrative manner; and so was obliged to get into the old beaten road as fast as possible.

But there is another rank of geometricians, between those, I have been speaking of, and Euclid: whom for distinction sake I shall call the *omnipotent*. These setting out with the supposition, that they can do every thing, undertake to demonstrate every thing. An *omnipotent* geometrician makes nothing of demonstrating the fifth proposition, for it is only supposing that he has cut the angle at the vertex in halves and the thing is done; never considering that the doing of this, requires it to be first proved that the angles at the base of an isosceles triangle are equal to one another. The only inconvenience of this method is, that should the author's omnipotency be once called in question; his whole fabric must be leveled with the ground.

But to return to *Clairaut*; I would have nobody imagine that I entertain a mean opinion of his abilities, from any thing that has been said: we only happen to differ as to the proper manner of communicating mathematical knowledge; which I contend should be either as facts or as demonstrations; and that there is no middle way; being persuaded that the mixing a kind of theory with the practice has been of bad consequence to both; as this has been the occasion of putting very improper persons into employments, by which many fine opportunities of making improvements have been lost to the world.

But farther *Clairaut's* method is even unnatural; for the science never could have been invented according to his plan; because it
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impossible that human reason, unless unnaturally confined, would be kept floating upon the surface of things : it is of the nature of man, when left to himself, to seek a solid foundation. His book, it is true, will make a superficial reasoner ; but mankind left to themselves, if they did so much, would necessarily do a great deal more.

And I now proceed to give an account of what seems to me a much more natural course for them to have taken in prosecuting their discoveries in this science ; which will at the same time explain the nature of Euclid's arrangement, and obviate every objection which has been made to it : but this shall be the subject of the next chapter.

C H A P. III.

The same subject continued.

It will be found to be a mistake to suppose that we naturally begin to examine the simplest things first ; for things are neither said nor done in the simplest manner by the first adventurers. Our attention is more like to be drawn to a subject by some particular circumstances, which made the consideration of it necessary, than by any determination of the mind to make it a matter of contemplation ; and by coming upon us thus, as it were unawares, our understanding is overpowered, when we undertake the examination of a thing encumbered with so many additional parts, which have no connexion with the subject in question ; and such as it will require leisure, skill and experience to separate from the others.

We are not therefore to imagine that the examination, either of *a line* or *a point* would be the first attempts for settling mens notions with regard to magnitude. It is more reasonable to suppose that the first kind of magnitude which would fall under the consideration of men, circumstanced as we are in society, would be rectilineal figures, and perhaps circles or even solids of some determinate shapes : not only because nature produces such a variety of them ready made to our hands, thus, as it were, inviting us to make them the subject of our mediation, but so soon as property
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in land became established, a man had only to discover that a greater quantity of it would yield a greater increase, to engage him to pay attention to this quantity ; either that he might be the better able to impose upon his neighbour, or to guard himself from the impositions of others.

And although they might at first imagine that the bounding lines would be sufficient for ascertaining their property ; yet they would come at last by reasoning and blundering, to perceive that the shape is necessary to be considered, as well as the bounding lines. This might lead them to make pictures or resemblances of their fields, by way of ascertaining their property, until by fresh experiments and blunders, it may be supposed they could perceive, that these representations, which I imagine to be made by the eye, without the assistance of other principles or instruments than such as occasionally offered themselves, were also inaccurate. Or even if this could have determined one's property, when joined to other circumstances upon the spot ; yet still a very material defect would be obvious ; as there would nevertheless be wanting, any thing like an accurate way of comparing different figures with one another, especially when their sides are numerous and their shapes unlike. And it is probable that they would at first drop all thought of being able to compare such with any degree of accuracy, but giving them up as unmanageable, would bend their whole attention to the simplest kind, very probably to the triangle ; until the great step was taken, which was to lay open the whole mystery of this science, by the man who first discovered that all rectilineal figures may be divided into triangles ; and this step once taken, they would then begin to tread on scientific ground. They would be made sensible of the use of the triangle, and exhaust all their thoughts and ingenuity to discover its properties.

But it should be observed, because I believe it to be the great spur to improvement, that there would probably appear great inequality in the accuracy of the different conclusions which their ingenuity would lead them to, when their curiosity was once awakened by so fair a prospect of being able to master their subject : some amounting to absolute demonstration, while a great many things perhaps fell short of such an evidence as was necessary to persuade

persuade them that the conclusions were true ; and such discordant parts would necessarily awaken their industry to give the whole an uniform appearance. It would be seen that if such a thing were proved such another would follow ; and they would collect as many of these consequences as they could relating to triangles ; what would prove them to be equal or unequal ; until they came to perceive the remarkable utility of this wonderful figure, and its aptitude to explain and connect the simplest and most intricate parts of the science.

And this seems to me to be the most natural account of the original and improvement of geometry ; because I am persuaded that if any thing, *even* uniformly inaccurate, had been produced by themselves ; or if the regular mechanical rules sufficient for the ordinary purposes of life had been communicated to them by some superiour Being ; mankind would have never thought of any farther improvement ; and the human race would have drudged on in the same beaten path void of curiosity and fully satisfied with their present acquisitions. But this we find to be a fact by experience ; for those who are instructed only in the practical parts of this science, proceed with as little hopes or desire of improvement, as the very instruments with which they work.

But to return to the subject ; let us suppose at first that they had it only in view to give irregular figures a more regular shape ; or in other words, that they wanted to perform the problem, which Euclid has given in the forty fifth proposition of his first book ; namely, to turn any rectilineal figure into a parallelogram having a given angle : it will be found that almost every proposition which goes before, is necessary towards a proper solution of this problem, and if the student would be at the pains to examine how far this is true in fact, it will assist him much in comprehending Euclid's arrangement ; especially if he attend particularly to the use which is made of the triangle : and with a few remarks upon this I shall conclude the chapter.

And first it is to be observed that straight lines and angles are too simple of themselves to bear a strict examination, with any hopes of discovering their properties ; and if we want to prove them to be either equal or unequal ; the lines must be sides of a triangle

and the angles are angles of some triangle ; the fourth, fifth, sixth, seventh and almost every proposition in the book furnishes us with instances to this purpose ; and it is owing to this sure guide deserting us that, we are brought into such difficulties by the eleventh common notion. It is true that by the assistance of a perpendicular, we seem to make a few steps in comparing angles which do not belong to any triangle, as in the thirteenth, fourteenth and fifteenth propositions, but as these depend upon the perpendicular, they are virtually supported by the triangle : and the same may be said where parallel lines are concerned. That the lines from the center to the circumference of a circle are equal, is the only principle by which lines can be proved to be equal independent of the triangle. Secondly we may observe how the transition is made from triangles to four-sided figures ; and the triangular spaces compared with parallelograms, by which a foundation is laid for turning all rectilinear figures, however irregular into parallelograms. And lastly when the usefulness of the triangle is considered, the reader will not be surprized to find so much pains taken, to settle every circumstance which can prove them to be equal, and in what cases some of their parts may be equal and the others not.

CHAP. IV.

Containing some remarks on the constructions in the first book.

IT has been remarked already that our author discovers great caution in the use of the postulates, by keeping all mechanical instruments as much as possible concealed from public view : and indeed he seems almost ashamed of them, for which reason he never constructs his problems upon a supposition that you make use of any particular instruments ; and this is necessary to be observed before we can form a proper judgement of the simplicity of his constructions. The common modern way of judging upon this point, proceeds upon a very mechanical principle indeed ; for the simplicity of a construction is to be determined by the number of times which you have occasion to open your compasses : but as
Euclid

Euclid knows nothing of a pair of compasses, it can never be supposed that this is the test to which he appeals ; but to the nature and number of ideas suggested to the mind by the different steps of the constructions. He does not care how often you are obliged to open your compasses, provided the perceptions, conveyed to the mind by his constructions, be the most simple and general which the circumstances of the case will admit of. Some particular instances will explain my meaning.

In the twenty third proposition we are required to make one angle equal to another ; and you save yourself the trouble of one opening of your compasses by making DEF an isosceles triangle : but Euclid requires you only to join EF ; which certainly is a simpler conception, than when you are moreover required to take care that your other two sides be equal ; and has also this great scientific advantage, that it is more general, by shewing that any triangle will answer the purpose ; when your mechanical construction goes upon the supposition that a particular kind of triangle is necessary. Thus it often happens that our compasses and understandings are at variance ; or rather our hand and head incline different ways ; what is most intelligible to the head, the hand finds the greatest difficulty in performing.

In the same manner, in the ninth proposition, three circles must be described according to Euclid's plan, whereas mechanically, which supposes that we can describe an isosceles triangle upon a given straight line, one opening of the compasses will answer the purpose ; and there is a like difference in the scientific and mechanical constructions of the tenth, eleventh and twelfth propositions.

The drawing of parallel lines Euclid uniformly refers to one principle ; and this problem is performed by making the alternate angles equal ; and this the nature of the science requires that he should do : however I have no objection to the use of any instrument which may answer the purpose, provided the student know what he is doing, and do not work at random without any settled principles.

Parallel rulers are of all contrivances, the most inconvenient for this purpose of drawing parallel lines ; but yet it might be of use

for the student to consider upon what principles we pretend to use them: and he will find that they are constructed upon geometrical principles, though not by any proposition in Euclid; for their parallelism depends upon the converse of the thirty fourth proposition, namely, that if the opposite sides of a quadrilateral figure be equal, they will be parallel; the demonstration of which will be obvious to the attentive reader; and he might at the same time consider the converse of the other part of the same proposition, viz. that if the opposite angles of a quadrilateral figure be equal, the figure is a parallelogram; this follows directly from this consideration; the opposite angles being equal to one another, and the four angles equal to four right angles, any two of them taken in order round the figure will be the half of the four right angles; that is equal to two right angles, and therefore the lines are parallel.

But the most convenient instrument for drawing parallel lines, as also a perpendicular, is a triangular piece of wood in the shape of a right angled triangle: for, by making this slide along the edge of a ruler, parallel lines may be drawn, upon the principle that the outward angle, is equal to the inward &c. by the twenty eight proposition. And if you have two parallel lines given you, a parallelogram may be formed by the thirty third proposition, by taking two equal parts of them, and joining their extremities which are towards the same parts; in short I would recommend every kind of construction which is derived from scientific principles, provided the reader give himself the trouble to find out the principles.

It sometimes happens that a direct construction will not answer the purpose; of which the amendment proposed by me in the construction of the forty fourth proposition is an instance; and if any one try to demonstrate the forty seventh proposition, by describing the square upon the same side of BC upon which the triangle ABC stands; he will meet with another instance of the same kind; for if he describe this square, he will find it no easy matter to prove that its sides are terminated in FG and KH or in these lines produced, which is absolutely necessary to make good the demonstration: but if he draw perpendiculars from the points B and C terminated in these lines and joining their terminations, it is easy
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to prove the figure to be a square; such indirect constructions should be particularly noted by the learner wherever they occur. In short to conclude what I have to say about constructions, I would have the student exercise himself, in them upon every occasion, or even to seek occasions for doing it, only with the caution to beware of contracting such prejudices as might corrupt his scientific principles, which can never be the case, if he know what he is about, and consider the nature and extent of his instruments.

CHAP. V.

Containing such remarks as may enable the reader to make a proper estimate of his progress in geometry when he has made himself master of this first book.

BEFORE he proceeds to the second book it will be proper for the reader to take a review of this first, not as a learner, but in order to make a proper estimate of his geometrical acquisitions: and he will not be much out in his reckoning if he judge of the whole by what he knows of the triangle. I have mentioned already that this figure is our great instrument of discovery, being equally necessary for determining the simplest properties of straight lines, and angles, as those of the most irregular figures.

The first general property which we learn of this figure is that all triangles are equal, which have two sides equal to two sides each to each, and the angles contained by the two sides equal; and that this equality does not simply hold with respect to the spaces, but is true of every single part of the triangle; their bases are equal, the two remaining angles in each are equal, viz. those under which the equal sides are extended, each to each: or in other words that the two sides, and the angle between them, fixes every part of the triangle without a possibility of change. The fifth and sixth propositions contain more partial properties of the triangle; by proving that the equality of the two sides will always insure the equality of the two angles at the base: and conversely that the equality of the two angles at the base fixes the equality of the
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the sides. Thus we know what will prove the triangle to be isosceles; and we know the most remarkable consequences which follow from its being such a triangle.

In the eighth proposition it is demonstrated that all triangles which have their three sides equal, each to each, are equal in every respect: or in other words, that the three sides fixes the triangle, so that neither its space or angles can possibly vary so long as the sides continue the same: and in the twenty sixth proposition we have two other tests to the same purpose: for if two angles and the side between them be given or fixt we cannot change any part of the triangle, neither the other two sides nor the other angle; so that all triangles which have these equal must be equal in every respect: or so long as two angles of a triangle and a side extended under one of them are fixt the whole triangle is fixt; therefore all triangles which agree in these particulars must be the same or equal in every respect.

These four tests which fix the triangles absolutely should be carefully examined not only separately, but compared together before the reader can make the proper use of them.

It is not only necessary to observe in what circumstances the triangles may be demonstrated to be equal; but it is of no less importance to settle in what cases, we can say, that the whole triangles or certain parts of them must be unequal. And first it appears, from the seventh proposition, that if the two triangles be upon the same base and upon the same side of it; their sides which are terminated in the same point cannot be equal. Also it follows from the twenty first proposition, that if the triangles be upon the same base, and the vertex of the one triangle within the other; their sides cannot be equal, either each to each or taken both together; but that the sides of the included triangle will be less than the two sides of the other.

Such conclusions as these indeed, we are both very dexterous and confident in drawing at random; and the following would appear strikingly conclusive; because it is proved in the fourth proposition, that, when the two sides are equal, each to each, and also the angle contained by the two sides, therefore the base is equal to the base; therefore undoubtedly we may infer from this, that

that if the included angles are not equal, neither will the bases be equal, but the one must be longer than the other; and certainly the larger angle ought, by the rule of right, to have the longer base extended under it. But softly, good sir, this is not geometrical reasoning: for although the equality and coincidence of the bases is demonstrated in this proposition when you keep up to the different parts of the supposition; yet it is not shewn that this is the only case in which that equality can happen: and indeed the changing a single circumstance will not prevent their equality for you may suppose AB to be unequal to DE and nevertheless the bases may be equal. So that however ready a superficial reader may be to adopt such conclusions, yet the geometrician will find that a great many steps are to be taken before he can demonstrate this to be true. Which is indeed done in the twenty fourth proposition, but after many necessary properties have been previously delivered. If the two sides are equal but the bases unequal it appears from the twenty fifth proposition, that the angle opposite to the greater base is the greater.

But farther, with respect to the angle: I have mentioned already, that we cannot compare angles unless we consider them as belonging to triangles; the sixteenth proposition is made use of to prove that one angle is greater than another: which is a very fundamental proposition and ought to be carefully examined: the simplest view which it is possible to take of it seems to me to be this; to consider the proposition as expressed thus; the angle ACD is greater than CAB and also the angle BCG is greater than ABC; which when properly considered will appear to be the same thing: and then, when we recollect that the angle ACD is equal to BCG, it may certainly be said that ACD is greater than either CAB or ABC: but this is a proposition which ought to be viewed in every light. By this proposition we likewise prove that lines in the situation, which the supposition requires in the twenty seventh can never form a triangle, and thus leads us to parallelograms; from the consideration of which we have a new and wonderful property of triangular spaces, namely, that if the base of the triangle continue the same or equal, you may vary the sides until they become a thousand times longer than in their first situation, and still the triangular

angular space will be the same. And this is also the principle by which we can change spaces from one shape to another as will be obvious to every one who understands the forty fifth proposition.

I shall conclude this dissertation, with a general admonition to the reader, to observe particularly the use of the supposition in every proposition; because I know nothing that is so easily neglected: the construction and even the references to other propositions may make a forcible impression upon the mind; but the supposition has hardly any thing which can awaken our attention to it, except a want of evidence in the conclusion, and there are those who have a wonderful inclination to lay aside their scruples at that part of a demonstration.

DISSERTATION IV.

WHOEVER has read the first book with care, will perceive the particular use of the triangle for discovering and connecting the different properties of magnitudes; and must at the same time be sensible of the uncommon genius of the author, who could apply this figure in such a manner as to make it discover its own properties, as well as those of other magnitudes. In his second book he presents us with a new instrument, the use of which is no less extensive than that of the triangle; and it is the purpose of this dissertation to draw the attention of the superficial reader to the properties and most obvious uses of the rectangle.

CHAP. I.

Of parallelograms.

TO understand this second book properly, it will be necessary to consider the thirty fifth proposition of the first book, in more points of view than are immediately presented to us by the proposition itself. It is there demonstrated that when the parallelograms are upon the same base, and between the same parallels, the parallelogram spaces are always equal: now this reduces an infinite variety of parallelograms to one single parallelogram as far as the space is concerned; and if a choice of a parallelogram is to be made, which is to represent all the others, the rectangle, or right-angled parallelogram ought to have the preference; because its bounding lines are the least, and its angle fixt or determinate; that its sides are the least will be obvious, by comparing a side of

the rectangle with the side of any other of this infinite variety of parallelograms just mentioned; as for instance AB with BE, the one of which subtends an acute angle and the other a right angle in the same triangle.

But this rectangle itself, is sufficiently or rather absolutely determined by any two of the lines about any one of its angles: and thus two straight lines, will fix an infinite variety of parallelogram spaces.

Now this is the first definition of the second book, which is thus expressed; Every right angled parallelogram, is said to be contained by any two of the straight lines containing the right angle. The space indeed is bounded by four straight lines; but not by four different straight lines, for the opposite sides are the same, or equal. Here the reader is carefully to observe that there is no difference made here between *equality* and *identity*; for whatever lines are equal, in the present case they are always considered as the same; that is if you make a right angled parallelogram, any two straight lines whatever, that are equal to the two lines about one of the angles; are said to contain the space, as well as the particular lines which make two sides of the very figure.

This circumstance perplexes beginners exceedingly; and ought to be made the subject of frequent meditation before one attempts to read a single proposition in this book. And here it would be proper to begin with considering the nature of a parallelogram, still more particularly: the reader has seen what will fix the parallelogram space: he is next to consider what will fix the whole parallelogram, sides and angles; and he will first observe that the two sides about any of the angles; will determine all the sides; because the other two sides are equal to these two; next let him try to find out what will determine the angles. Now any angle of a parallelogram being given, it is impossible to change any of the others; but this the reader will do well to demonstrate; which he may do as follows; any two angles of a parallelogram which follow each other in order are equal to two right angles, one of these being fixed surely the other is fixed also; and the opposite angles of parallelograms are equal; therefore it is obvious that one angle of the parallelogram fixes all the others. The reader should demonstrate

monstrate this formally and then he will see that the parallelogram is entirely fixed when an angle, and the two sides about that angle are given. But as I know from experience that the student will not perceive this immediately; let him take only the two sides, and the angle between them; and with these compleat a variety of parallelograms; and he will always find them the same. Not that this construction is necessary towards a right understanding of the conclusion; but I know that it is necessary for fixing the reader's attention; and without attention it is impossible to comprehend any conclusion. Now this point being once settled, it is very clear, that if the parallelogram be right angled, the two sides about one of the angles, or even any lines, equal to these, will fix the parallelogram; from which the full import of this definition may be understood.

CHAP. II.

Containing some remarks on the principles made use of in demonstrating the first eight propositions.

THE simplicity of the demonstrations made use of in the first eight propositions, will be readily acknowledged by every one who understands them: and yet it often happens that the student is at a loss to comprehend their meaning, especially upon the first reading; which is owing to two circumstances; first because he does not understand the first definition; and secondly because we have no common notions of the properties of figures contained in this book; if some property of a triangle should be accidentally mentioned to a person ignorant of the principles of this science, he would nevertheless form some opinion concerning its truth or falsehood from some common notions of his own; but read the enunciation of the third proposition in this book to him; and he will have no more opinion of its truth or falsehood than if it were delivered to him in an unknown tongue. And supposing these two difficulties to be got over; that is supposing the first definition to be well understood, and the meaning of the proposition become familiar, it is impossible for any evidence to present itself to the

mind in a simpler form, than that which these demonstrations offer; as the progress of it in general is thus; you describe a certain rectangle or square, which is divided into right angled parallelograms or squares by lines drawn parallel to the sides; and the whole mystery consists in settling what these spaces are according to the first definition. For instance, in the first proposition, as the whole is equal to its parts, the rectangle, or right angled parallelogram, BH is equal to the rectangles BK, DL, EH taken all together; so that we have only to settle what the rectangles BH, BK, DL, EH are according to the first definition, and the demonstration is complete. In the same manner in the second proposition (which by the bye is only a particular case of the first,) the square ABED is made up of the two rectangles AF and CE; so that we have only to determine what these are according to the first definition, and then the thing proposed is demonstrated. Again in the third proposition, the rectangle AE is made up of the rectangle AD and the square CE; so that we have only to determine what these are according to the same first definition, and the proposition is demonstrated and so in other instances.

The first eight propositions seem to a superficial reasoner to rest in some measure upon a kind of intuitive evidence; because the form of demonstration differs remarkably from that made use of in the remaining propositions of this book; in which the conclusions are inferred from the forty seventh proposition of the first book without exhibiting the squares. But Euclid never draws any consequence from what we are supposed to see, though we ourselves may: for his demonstrations are equally conclusive whether the figures upon which we reason be the objects of our senses or not. The different spaces exhibited in these figures arise from the construction, and have their existence and properties from that, whether we see them or not.

C H A P. III.

Of the addition and subtraction of rectangles and squares.

WHEN the learner has made himself master of the demonstrations contained in this second book; it will be proper to turn his thoughts

thoughts to the use which may be made of the properties of these rectangles and squares which he has been considering: and as almost the simplest use, which can be made of magnitudes, is adding them to one another or taking them from one another; it is very manifest that these properties will be in a manner useless, until we have acquired a readiness in adding them together, and taking them from one another: and this indeed is one of the most intricate and extensive principles in geometry. Much of this business may, and indeed ought to be learnt from this very book; for whoever should treasure up in his memory whatever is contained in this book, it would be but useless lumber unless considered under this particular point of view. And so much for the necessity of this practice; but a few examples will best explain my meaning.

Suppose in the second proposition you were required to add the rectangle contained by AB and BC to the rectangle contained by AB and AC; these taken together make the square of AB: again if from the square of AB you take away the rectangle contained by AB and BC; the remainder is the rectangle contained by AB and AC; all this is obvious from the inspection of the figure; but as these consequences may be wanted when the figure is not at hand for inspection, the reader ought to be able to derive them readily from any single straight line cut into two segments. But again in the third proposition if you add the rectangle contained by AC and CB to the square of CB it makes the rectangle contained by AB and BC; or if you add the rectangle contained by AC and CB to the square of AC it makes the rectangle contained by AB and AC: And farther if from the rectangle contained by AB and BC you take away the square of BC there remains the rectangle contained by AC and CB.

In the fourth proposition if you take the squares of AC and CB from the square of AB there is left the rectangle contained by AC and CB taken twice: or if the square of AC be added to the rectangle contained by AC and CB taken twice; their sum will be equal to the difference between the squares of AB and BC; and this is also obvious from the inspection of the figure.

The fifth proposition, in which is compared the rectangles made by the equal segments of a line with those made by the unequal segments,

segments, furnishes frequent opportunities for this kind of practice; thus if from the square of half the line BC you take away the square of CD the remainder is the rectangle contained by AD and DB; and this square of CD added to the rectangle contained by AD and DB makes up the square of half the line. And in the sixth proposition if from the square of CD you take the square of CB the remainder is the rectangle contained by AD and DB. One might just observe upon the fifth; that the rectangle contained by AD and DB and the square of BC are bounded by the same extent of line; but that the spaces inclosed differ by the square of CD.

It would be tedious to be more particular upon this subject; I shall therefore conclude with observing that squares are generally added together or subtracted from one another by the forty seventh proposition of the first book; and that they are doubled by the assistance of a right angled isosceles triangle as in the ninth and tenth propositions; and may be halved upon the same principles.

CHAP. IV.

In which is shewn the absurdity of applying numbers to illustrate the propositions in this book.

IF you draw two indefinite straight lines at right angles to each other, and cut off from one, seven equal parts, beginning at the angular point, and from the other four of the same equal parts, and compleat the right angled parallelogram; then, through each of the divisions, drawing lines parallel to the sides of the rectangle; the whole surface is divided into squares all equal and twenty eight in number, which is the product of four multiplied by seven. And because this parallelogram is equal to any other upon the same or an equal base and between the same parallel lines; and a triangle in the same circumstances, the half of it: therefore this other parallelogram, though not divisible into squares, is said to contain twenty eight such squares; and the triangle fourteen of the same. And as any rectilineal figure is divisible into triangles, upon these principles

principles any rectilineal figure may have its contents exprest in square measure, inches feet or yards according to the measure by which you suppose your lines to be divided.

Now this way of expressing the surface or contents of a right angled parallelogram in square measure by the product of the two sides about one of the right angles, has introduced a very absurd *practice* upon this second book, which some facetious gentlemen have been pleased to stile illustrations.

But we shall be best able to decide how far this method of *operation* can be called an illustration by considering the use for which it is intended. Now the common affairs of life require, that we should pay a particular attention to small and large portions of extension within certain limits; which made it necessary to assume particular parts of extension as a common measure; and for this purpose at first certain parts of the human body seemed to have been used as is obvious from the names of measures; an inch, a foot, a span &c. and very probably he who observed that the feet of different men were unequal in length, and thus proved the necessity of having some fixt standard measure, stood pretty high in the opinion of his cotemporaries for ingenuity. Suppose them to have made the standard measure which we call a foot; for some purposes this would be too large and for others too small. The distance of one place from another exprest in feet could bring no distinct idea to the mind; it would be an improvement to take three feet and make an unit of that, under the different name of a yard, but still more to take one thousand seven hundred and sixty yards and call it a mile. And by such contrivances our perceptions of distance might be made tollerably accurate and suited to our circumstances. Surfaces are measured upon the same plan, and exprest in square feet, yards or miles upon the principles explained above: and to the great comfort of the unthinking part of mankind all surfaces may be compared in a manner sufficiently accurate for the purposes of common life, not only without our ever attending to a surface; but what is more without our having any notion of extension. For our attention is turned to no particular kind of magnitude, as we are only to consider whether one number be greater than another; and here it is not necessary to form your judgement
by

by the nature of numbers, but by a particular method of notation which you have been taught.

It would be strange enough to put one upon measuring such figures as Euclid treats of in this second book, in order to get at those properties which he says belong to them; but it would be the greatest absurdity to call this an illustration of Euclid's demonstrations; as the two methods are founded upon principles, in a certain sense directly contrary to each other: these mensurations drawing the reader's attention from Euclid's plan to fix it upon something else; or, more properly speaking, upon nothing. Neither is the conclusion scientifically accurate; for the lines taken at a venture cannot be divided according to any measure; particularly if the line be cut as is required in the eleventh proposition, it is impossible to measure it and its parts. But to rest this matter entirely upon the most material objection, viz. that it is directly contrary to Euclid's ideas. He teaches how to turn any rectilineal figure into a square in the last proposition of this book. Which shews that his plan is to make every step we take the object of the understanding, and therefore does not present a multitude of things to the mind which it is impossible for it to comprehend, but two distinct things for its contemplation: he does not express irregular figures by a multitude of small squares of any particular name; but reduces any two, which he may have occasion to compare, to two squares; nor has his reasoning a reference to any particular measure. In short if my intention was to purchase figures, I might trust to this practice; but if I meant to reason about their properties it must be entirely laid aside.

DISSERTATION V.

THE triangle and rectangle two of the great instruments of geometrical investigation, being considered, Euclid proceeds to the third, which is of no less importance than either of the other two. And is here in its proper place, because the properties of the circle are derived, partly from the triangle, and partly from the rectangle.

In the second book the propositions are arranged according to their simplicity; and not as in the first book according to the dependence which the propositions have upon each other: because the tenth might be read first, for any connexion which it has with the first nine propositions; the four last indeed are connected with some of the preceding propositions, but none of the others. In this third book the arrangement is made partly according to the simplicity of the properties and partly according to the connexion which they have with one another; and the explanation of this arrangement with some remarks upon the method of demonstration shall be the subject of this dissertation.

CHAP. I.

Containing remarks upon the arrangement of the propositions.

IN the two first books our author considers the circle only as a mechanical instrument, and the use made of it, rests entirely upon the third postulate; and unless it were introduced upon a different footing, it could hardly be reckoned a geometrical figure. In short in the first two books it is only a pair of compasses with an indefinite extent and perfectly accurate. But here in this book

it puts on a very different character, and is a geometrical figure bounded by one line called the circumference; and it is also added that there is a certain point within the figure from which all the lines drawn to this circumference are equal; but this point is to be found, before we can expect to make much of the properties of the circle. The first proposition teaches us how to find a straight line passing through this point; and not simply a straight line, but a finite straight line, the cutting which in halves will find the center. This being found it seems both natural and necessary to consider whether the circumference of a circle be totally and essentially different from a straight line; which it is proved to be in the second, by shewing that every straight line joining any two points in the circumference falls entirely within the circle, and can neither apply itself to any part of the circumference, nor even meet it except in the two assumed points; from which it follows that a straight line cannot cut the circumference of a circle in more points than two.

But to return to the center; the investigation of this point would shew that the center will always be found in any straight line cutting another, terminated by the circle, in halves and at right angles: this discovery brings us directly to the third proposition; for it would not be difficult to imagine that any straight line drawn from the center to the point in which any line was cut in halves, would fall in with this perpendicular, and the contrary; and this would lead to the demonstration of the fourth proposition, which must have been suggested already by the property of the center, *that all lines passing through it and terminated by the circle are cut in halves there*; because it would be proper to demonstrate that no other point has this property, and it is here shewn that not even two lines terminated by the circle can be cut in halves in any other point. But not to be tedious, the meditating upon the center would very naturally lead to the discovery of the first fifteen propositions; and it is hardly to be imagined that a geometrician would drop the contemplation of the second proposition before he had discovered in what manner a straight line drawn through any point in the circumference of the circle would meet its circumference, which would lead to the consideration of the sixteenth, seventeenth, eighteenth

eighteenth and nineteenth propositions. And it would be very proper for the student to examine the first nineteen propositions again and again, before he proceed any farther in the book.

CHAP. II.

The same subject continued.

THE first and second propositions might lead in a very easy and natural manner to the discovery of the first nineteen propositions: but the idea of similar segments is by no means an obvious one; for the common notion is to the last degree vague and undeterminate: so that ever since I could pretend to form any opinion upon this subject, I have always regarded this as one of the most masterly parts of geometry, of the most subtile invention, and as having the most extensive and unexpected consequences; and yet when it is once considered that the angle at the center is double of the angle at the circumference, it all follows in a simple and seemingly obvious manner; for when it is once proved that all angles in the same segment are equal; if they should once begin to conjecture that these angles did not depend upon any particular circle, but were fixt by the segment of the circle which they were considering, they would then arrive at their definition of similar segments, and the whole business was done: and segments of circles could be compared with each other not in a vague manner but accurately and with the same precision as rectilineal figures; and the equality of the circumferences of equal circles could be ascertained in a manner full as satisfactory as the bases of the two triangles are proved to be equal in the fourth proposition of the first book: and thus we could take one circumference double or treble of another, with the same ease that one straight line may be made double or treble of another. It is not easy to say positively, what led them to their definition of similar segments, but the most natural step to it seems to be the proving that the angles in any semicircle are right angles, and that there seemed to be no other circumstance about a segment fixt except the angles which it contained. Those who judge of the demonstration of a proposition by the number

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of

of steps which it contains are apt to pass over the demonstrations of some of these propositions, as of little difficulty and less importance; but to the judicious they will appear, as they really are, of the first importance in this science. The learner who would be properly acquainted with this subject should consider the propositions from the nineteenth to the thirty fifth, as all tending to the same purpose, and after giving them a very particular separate examination, carefully trace their connexion with each other.

In the thirty fifth proposition it is said that the rectangles contained by the segments of lines intersecting one another within a circle are always equal; and the thirty sixth is, that the square of any tangent is equal to the rectangle contained by the whole cutting line, drawn from the same point, and the segment of it without the circle. From which it follows that if any number of straight lines be drawn from a point without a circle, cutting the circle, the rectangles contained by the whole line and the part without the circle are all equal, which is the same property as that delivered in the thirty fifth, when the lines intersect one another within the circle.

To conclude, the reader will perceive upon a careful examination of this book, that there are three great sources from which the properties of the circle are derived, namely the center; the similar segments; and the equal rectangles made by the segments of lines terminated by it. We may say indeed that they are all ultimately derived from the center; but it appears to me that the student will find it very useful to consider them under these three distinct heads.

The plan of the fourth book is so very obvious and regular that it seems sufficient just to advertise the reader, that it is taken up in performing four problems, inscribing the figure within the circle describing the figure about the circle, inscribing a circle within the figure and describing a circle about the figure.

C H A P. III.

Containing some remarks on the demonstrations.

EUCLID has a certain idea of demonstration which he invariably adheres to; and although it be perfectly regular and accurate, yet

yet it does not always take in every circumstance which beginners require to be put in mind of: this I mention to avoid the imputation of vanity, lest any additional hints to fit the demonstration more to the disposition of beginners might be looked upon as an attempt to supply some defect in the demonstration: whereas they are intended only to remind the reader, that he may mistake the meaning of the demonstration, by trusting to the first view which is presented to his imagination. And I here again repeat, what has been mentioned before, that he should examine the simplest instances first, making them gradually more general until he arrive at such a conclusion as the nature of the thing requires, observing what consequences have been drawn from every part of the supposition, and whether any consequence depends upon a particular position of the lines; and this is to be done not from any hopes of correcting the author, but only to get the better of his own prejudices; because the author's task would be endless and indefinite, if he should undertake to give such demonstrations as would remove all the prejudices which we are apt to take up upon this subject.

In the second book, there is something so determinate in the position of the lines, that any variation which the figures can undergo, either in shape or size, will hardly occasion any misconception in the meaning of the proposition. But this is so far from being the case with regard to the *figure* which is the subject of this book, that by enlarging its size it may seem to have very different properties: for a circle described only with a *radius* equal to the semidiameter of the Earth, will differ so little from a straight line in its circumference, according to vulgar comprehension, that perhaps for an hundred yards it would make a more accurate straight line than was ever drawn by any instruments: and how are the contacts, intersections and other properties of such circles to be determined but by reasoning scientifically from the nature of the circle. But neither the necessity nor beauty of such demonstrations can be perceived by those who confine their notions to such circles as they can describe with a pair of compasses.

I have indeed humoured this kind of prejudice in the description of the figures for the second and sixteenth propositions, because I never found a beginner who could enlarge his notions of these figures to make the position which the demonstration requires seem probable.

probable. The reader may therefore give these demonstrations a turn more suitable to the figures by considering that when he has proved, in the second proposition that DB is longer than DE and that they are both drawn from the center, and as DB reaches only to the circumference, therefore the shorter line DE cannot extend so far. In the same manner, in the sixteenth DA being proved to be longer than DG ; and being both drawn from the center, and DA reaching only to the circumference the line DG cannot reach so far; therefore the point G is within the circle, and consequently the line AF cuts the circle. Let no one imagine that I propose this as a demonstration; I have only expressed myself thus in compliance with vulgar prejudice; for the legitimate demonstrations are those given by Euclid.

In the eighth proposition the reader might just observe, that DK may be proved to be shorter than DL ; in the same manner that DF in the former part of the proposition is proved to be less than DE ; and it might be farther proper to compare this proposition with the seventh; when it will appear that they both contain the same property of the circle, only the point is taken within the circle in the one and without it in the other; also it may be proper to remark that the point D must be in the plane of the circle, otherwise the line DM would not cut the circumference; and the same is to be observed when we are required to draw a tangent from a point without a circle: and that the tangent itself must be in the plane of the circle. But Euclid takes it uniformly for granted that all his lines are in the same plane until he comes to the eleventh book. In the ninth proposition the reader might suppose the point E to be within the angle ADB and observe what difference it will make in the demonstration; he cannot then prove that DC will be greater than DB and DB than DA ; but he can prove that DB is greater than DC which is sufficient for his purpose. It is possible to mistake the meaning of the eleventh proposition, as if it confined the contact of the circles to a single point; but that is not its meaning; suppose any point of contact, the straight line joining the centers will pass through the very point which you take; for if the line FG was also to pass through another point of contact, the absurdity would be just as great, that GD should be equal to
GH

GH and at the same time greater, as that it should be less than GH and at the same time greater. There is something rather singular in the eighteenth proposition, for it seems to me to be nothing but the corollary to the sixteenth proposition; because a tangent there; is a straight line at right angles upon the extremity of the diameter; according this notion therefore the one proves that AE is perpendicular to AB; and the other that AB is perpendicular to AE which is certainly the same thing. Unless it should be urged in favour of Euclid that the corollary does not say that every tangent is at right angles to the diameter passing through the point of contact, but only if it be at right angles it will be a tangent; but it appears to me that there is no room for this distinction in the present case; for when we say that AE is perpendicular to AB; it is not the converse of that proposition to say that AB is perpendicular to AE. It would be a proper exercise for the student in reading the seventeenth proposition to prove that two tangents might be drawn from any point without a circle, and that they are equal, though this is not according to Euclid's plan.

I come now to the twentieth proposition, which should be examined with the greatest care, by describing figures for every distinct case, and varying the position of the angular point through the whole circumference BADC: after this is done it will be proper for the student to make an obtuse angle at the circumference, which will give him a particular kind of angle at the center, and it might admit of some dispute whether it is to be reckoned an angle in Euclid's sense of the word; however it may be proved to be double of the obtuse angle at the circumference of the circle. As this is a very fundamental proposition, the reader may just observe, (what I have recommended to him for a constant practice) the consequences which are drawn from every part of the supposition; the angles are said to be at the center and circumference; and to stand upon the same circumference. Now if either of these circumstances be omitted, the triangles will not be isosceles &c. The next proposition should also be particularly examined in all its cases; when the segment is greater than a semicircle, equal to a semicircle, and less than one; indeed if an angle increasing until the lines take the same direction is to be considered as equal to two right

right angles, and if what an angle wants of four right angles is to be regarded as an angle, there is but one case of the proposition; however if these do not make an angle, yet by drawing a line through the center they will be angles in Euclid's sense of the term, and the demonstration of the other two cases will be obvious.

I shall conclude my remarks upon this book by desiring the reader to take notice of a theorem which follows from the thirty third proposition; viz. If two triangles have their bases equal, and also the angles under which the bases are extended; and if they be between the same parallel lines; the two triangles will be equal in every respect. This is mentioned to shew that the joining the properties of the circle to those of the triangle will enlarge our notions very much; because we never could have discovered this by the triangle alone.

In the fifth proposition of the fourth book, *Simson* thinks that Euclid should have proved that the perpendiculars to the sides of the triangles will meet; and it seems to me equally necessary to shew that the tangents will meet in the third proposition; indeed if they did not the angle AKB would cease to be an angle, and AKB would be a straight, but it is equal to the outward angle of a triangle therefore &c. Or in both these instances it may be demonstrated as *Clavius* has done by joining AB in the one; and DE in the other; which brings them to the eleventh common notion.

But although I recommend it to every student to examine all possible suppositions, yet for reasons which have been mentioned already such omissions seem to me very consistent with Euclid's plan of demonstration; for he supposes his reader attentive; and therefore gives him only, what he judged to be the necessary assistance.

97

T H E
E L E M E N T S
O F
E U C L I D.
B O O K I.

D E F I N I T I O N S.

- I. **A** POINT is that of which no part *can be taken*. 2. But
A LINE *has* length without breadth. 3. Also extremi-
ties of a line *are* points. 4. Any *line* which lies evenly
between the points in itself is A STRAIGHT LINE. 5. And that
is A SURFACE which has length and breadth only. 6. Also extre-
mities of a surface, *are* lines. 7. Any surface which lies evenly
between the straight lines in itself is A PLANE. 8. But A PLANE
ANGLE is the inclination of lines to one another; *that is* of two
lines in a plane, meeting each other and not lying in a straight
line. 9. Also it is called A RECTILINEAL ANGLE, when the
lines containing the angle are straight. 10. But when a straight
line standing upon a straight line makes the adjacent angles equal
to one another; each of the equal angles is A RIGHT ANGLE: and
the standing straight line is called A PERPENDICULAR *to that* on
Vol. I. A which

Book I. which it stands. 11. AN OBTUSE ANGLE is that which is greater than a right angle. 12. But AN ACUTE *angle* is that which is less than a right angle. 13. That which is an extremity of any thing is *called* A TERM. 14. The *space* bounded by one or more terms is called a FIGURE. 15. A CIRCLE is a plane figure bounded by one line, which is called a CIRCUMFERENCE: upon which all the straight lines falling, from ONE POINT of those lying within the figure, are equal to one another. 16. And the point is called the CENTER of the circle. 17. But a DIAMETER of a circle is any right line drawn through the center, and terminated both ways by the circumference of the circle: which also divides the circle into two equal parts. 18. Also A SEMI-CIRCLE is the figure bounded by a diameter, and the circumference of the circle cut off by it. 19. A SEGMENT of a circle is *the figure* bounded by any straight line and the circumference of a circle. 20. The *figures* bounded by straight lines are *called* RECTILINEAL FIGURES. 21. Of *these*, such as are *bounded* by three, are *called* TRILATERAL. 22. As those by four QUADRILATERAL. 23. But those bounded by more than four straight lines *are called* MULTILATERAL. 24. Again of trilateral figures, AN EQUILATERAL TRIANGLE particularly is that which has three equal sides. 25. But an ISOSCELES, that which has only the two sides equal. 26. And a SCALENE, that which has the three sides unequal. 27. But moreover of trilateral figures, that again is a RIGHT-ANGLED TRIANGLE, which has a right angle. 28. And an OBTUSE ANGLED *triangle*, that which has an obtuse angle. 29. Also that which has three acute angles, an ACUTE-ANGLED *triangle*. 30. But of quadrilateral figures, that is a SQUARE, which is equilateral and rectangular. 31. And an OBLONG, that which though rectangular is not equilateral. 32. And a RHOMBUS, though equilateral, is not rectangular. 33. But that which is neither equilateral, nor rectangular, having only its opposite sides and angles equal to one another, *is* a RHOMBOIDES. 34. And *all other* quadrilateral figures except these may be called TRAPEZIUMS. 35. Any straight lines, which are in the same plane, and being produced indefinitely towards both sides meet each other on neither, are PARALLELS.

P O S-

P O S T U L A T E S.

1. Let it be taken for granted, that a straight line may be drawn from any one point to any other point. 2. And that a finite straight line may be produced in a straight line continually. 3. Also that a circle may be described with any center and *at any distance*.

C O M M O N N O T I O N S.

1. Magnitudes, which are equal to the same magnitude, are equal to one another. 2. And if equal ones be added to equal ones, the wholes are equal. 3. And if from equals, equals be taken away, the remainders are equal. 4. And if to unequals, equals be added, the wholes are unequal. 5. Also if from unequals, equals be taken away, the remainders are unequal. 6. And the doubles of the same are equal to one another. 7. Also the halves of the same are equal to one another. 8. Magnitudes which fit each other exactly are equal to one another. 9. Also the whole is greater than its part. 10. And all right angles are equal to one another. 11. And if a straight line meeting two straight lines, make those angles, which are inward and upon the same side of it, less than two right angles, the two straight lines being produced indefinitely will meet each other on the side where the angles are less than two right angles. 12. Also two straight lines do not inclose a space.

P R O P O S I T I O N I.

Upon a given finite straight line to describe an equilateral triangle.

Let AB be the given finite straight line: it is required to describe an equilateral triangle upon the straight line AB.

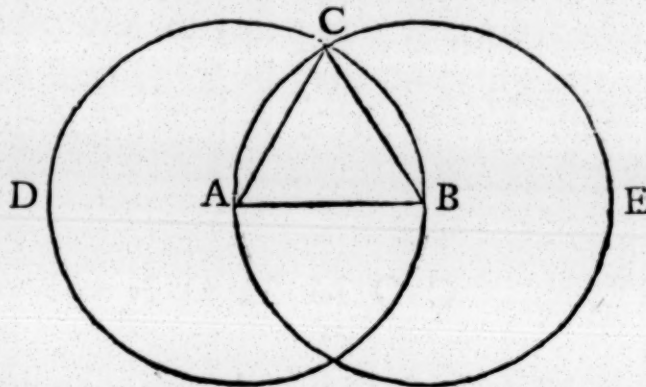
With the center A and at the distance AB (by post. 3.) let the circle BCD be described: and again, with the center B, but at

A 2

the

Book I. the distance BA (by post. 3.) let the circle ACE be described: and from the point C in which the circles cut one another (by post. 1.) let the straight lines CA, CB, be drawn to the points A and B.

Since therefore the point A is the center of the circle BCD (by Def. 15.) AC is equal to AB: again, because the point B is the center of the circle ACE (by Def. 15.) BC is equal to BA. But it has been also demonstrated that CA is equal to AB; each



therefore of the *straight lines* CA, CB is equal to AB; but (by com. not. 1.) magnitudes which are equal to the same magnitude are equal to one another, and the *straight line* CA is therefore equal to the *straight line* CB; wherefore the three straight lines CA, AB, BC are equal to one another.

The triangle ABC is therefore (by Def. 24.) equilateral, and is described upon the given finite straight line AB. Which was to be done.

P R O P. II.

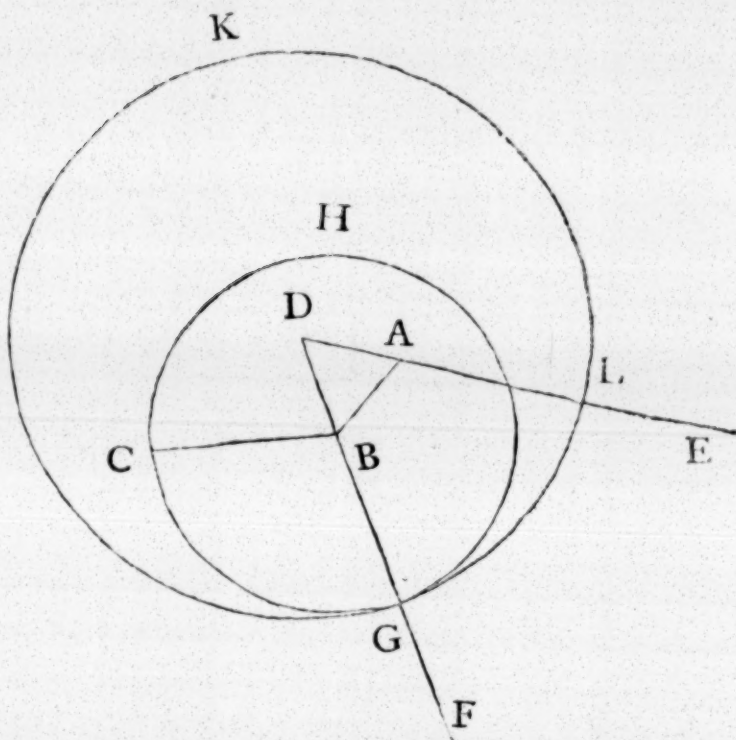
At a given point, to place a straight line, equal to a given straight line.

Let the given point be the *point* A, and the given straight line the *straight line* BC: it is required at the point A to place a straight line equal to the straight line BC.

For let the straight line AB be drawn (by post. 1.) from the point A to the point B: and upon it let the equilateral triangle DAB be described (by prop. 1.) and let the straight lines AE, BF be produced (by post. 2.) in straight lines with DA and DB; and then with the point B as a center and at the distance BC, let (by post. 3.) the circle CGH be described: and again, with the point D as a center, and at the distance DG let (by post. 3.) the circle GKL be described.

Since

Since therefore the point B is the center of the circle CGH the straight line BC is (by def. 15.) equal to BG: and again, because the point D is the center of the circle GKL the straight line DL is equal to the straight line DG (by Def. 15.) the parts of which viz. the straight line DA is equal to the straight line DB, therefore the straight line AL the remainder is equal (by



com. not. 3.) to the straight line BG the remainder, but the straight line BC has been shewn to be equal to the straight line BG: each therefore of the straight lines AL, BC is equal to the straight line BG; but magnitudes equal to the same, are also equal to one another: and therefore the straight line AL is equal to the straight line BC.

Wherefore at the given point A, the straight line AL is placed equal to the given straight line BC. Which was to be done.

PROP. III.

Two unequal straight lines being given, to cut off a part from the greater equal to the less.

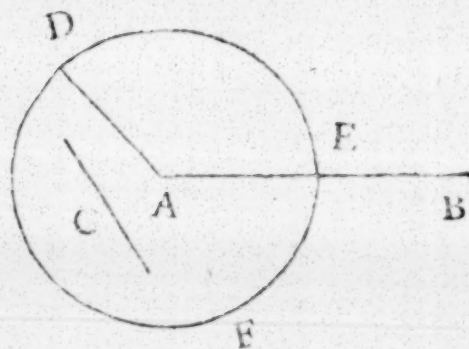
Let the straight lines AB and C be the two unequal straight lines given: it is required from the greater the straight line AB to cut off a straight line equal to the straight line C the less.

Place (by prop. 2.) at the point A a straight line AD equal to the straight line C; and then with the point A for a center, but at the distance AD, let the circle DEF be described.

And

Book I.

And since the point A is the center of the circle DEF, the straight line AE is equal to the straight line AD (by def. 15.) but the *straight line* C is also equal to the *straight line* AD: each therefore of the *straight lines* AE and C is equal to the *straight line* AD: wherefore also the *straight line* AE is equal (by com. not. 1.) to the *straight line* C.



Wherefore two unequal straight lines AB and C being given, from the *straight line* AB the greater, the *straight line* AE has been cut off equal to the *straight line* C the less. Which was to be done.

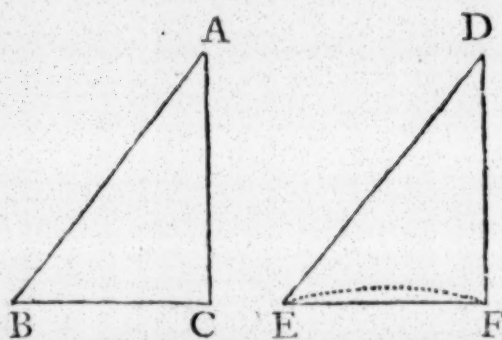
PROP. IV.

If two triangles have the two sides equal to the two sides, each to each, and have the angle equal to the angle, the *angle* contained by the equal straight lines: they shall also have the base equal to the base, and the triangle shall be equal to the triangle; and the other angles under which the *two* equal sides are extended, shall be equal to the other angles, each to each.

Let the *triangles* ABC, DEF be two triangles having the two sides the *straight lines* AB, AC equal to the two sides the *straight lines* DE, DF each to each, the one, the *straight line* AB to the *straight line* DE; the other the *straight line* AC to the *straight line* DF; and an angle, that contained by BA, AC or BAC equal to that contained by ED, DF or EDF: I say, that also the *straight line* BC a base is equal to the *straight line* EF a base: and the triangle ABC shall be equal to the triangle DEF, and the other angles, under which the *two* equal sides are extended, shall be equal to the other angles each to each; the one, the angle contained by AB, BC or ABC to that contained by DE, EF or DEF; the other, that contained by AC, CB or ACB to that contained by DF, FE or DFE.

For

For the triangle ABC being applied to the triangle DEF; and the point A in particular being put upon the point D, but the straight line AB upon the straight line DE, also the point B will apply itself to the point E,



because the straight line AB is equal (by supposition) to the straight line DE: but the straight line AB having applied itself to the straight line DE, the straight line AC will also apply itself to the straight line DF, because the angle contained by BA, AC or BAC is equal to that contained by ED, DF or EDF (by supp.): wherefore also, because again (by supp.) the straight line AC is equal to the straight line DF, the point C will apply itself to the point F: certainly also the point B has applied itself to the point E, wherefore the straight line BC a base, will apply itself to the straight line EF a base: for if, the one, the point B having applied itself to the point E, and the other, the point C to the point F, the straight line BC a base will not apply itself to the straight line EF, two straight lines will inclose a space, which (by com. not. 12.) is impossible: therefore the straight line BC a base will apply itself to the straight line EF, and (by com. not. 8.) will be equal to it: wherefore also the whole, the triangle ABC will apply itself to the whole, the triangle DEF, and (by com. not. 8.) will be equal to it: and the other angles will apply themselves to the other angles, and (by com. not. 8.) will be equal to them; the one contained by AB, BC or ABC to that contained by DE, EF or DEF; the other contained by AC, CB or ACB to that contained by DF, FE or DFE.

If therefore two triangles have the two sides equal to the two sides, each to each, and have the angle equal to the angle, the angle contained by the equal straight lines: they shall also have the base equal to the base, and the triangle shall be equal to the triangle; and the other angles, under which the two equal sides are extended, shall be equal to the other angles, each to each. Which was to be demonstrated.

P R O P.

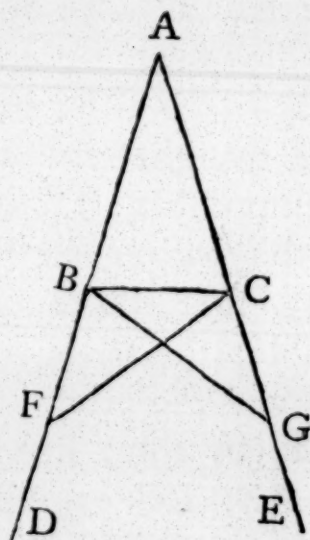
PROP. V.

The angles at the base of isosceles triangles are equal to one another; and if the equal straight lines be produced, the angles under the base shall be equal to one another.

Let the *triangle* ABC be an isosceles triangle, having the side AB equal to the side AC , and let the straight lines BD , CE be produced in straight lines to AB , AC ; I say that the one, the angle contained by AB BC or ABC is equal to the angle contained by AC , CB or ACB ; the other, contained by CB , BD or CBD to the angle contained by BC , CE or BCE .

Let any point which you may accidentally happen upon, as the point F be taken in the line BD , and let the straight line AG be cut off (by prop. 3.) from the greater the straight line AE , equal to the less the straight line AF ; and let the straight lines FC , GB be drawn.

Since therefore the one straight line AF is equal to the straight line AG and the other AB to the straight line AC , certainly the two FA , AC , are equal to the two, GA , AB , each to each, and they contain a common angle, the angle contained by FA , AG or FAG : therefore (by the 4th prop.) the straight line CF a base is equal to the straight line GB a base, and the triangle AFC will be equal to the triangle AGB , and the other angles will be equal to the other angles, each to each, under which the equal sides are extended, the one contained by AC , CF or ACF to the angle contained by AB , BG or ABG , and the other contained by AF , FC or AFC to the angle contained by AG , GB or AGB .



And since a whole the straight line AF is equal to a whole the straight line AG , parts of which the straight line AB is equal to the straight line AC , wherefore (by com. not. 3.) the straight line BF a remainder is equal to the straight line CG a remainder: but the straight line FC has been proved equal to the straight line GB ; certainly

tainly the two BF, FC are equal to the two CG, GB, each to each; and an angle the *angle contained* by BF, FC or BFC is equal to an angle, the *angle contained* by CG, GB or CGB, and the *straight line* BC is a common base of them. And therefore (by the 4th prop.) the triangle BFC will be equal to the triangle CGB, and the other angles, under which the *two* equal sides are extended, will be equal to the other angles, each to each; wherefore the one *contained* by FBC is equal to the *angle contained* by GCB; and the other *contained* by BCF to the angle contained by CGB. Because therefore a whole the angle *contained* by ABG has been proved equal to a whole the angle *contained* by ACF, *parts* of which, the *angle contained* by CBG is equal to that *contained* by BCF, therefore the *angle contained* by ABC which remains is equal (by com. not. 3.) to that *contained* by ACB which remains, and they are at the base of the triangle ABC; but the *angle contained* by FBC has been also proved equal to that *contained* by GCB, and they are under the base.

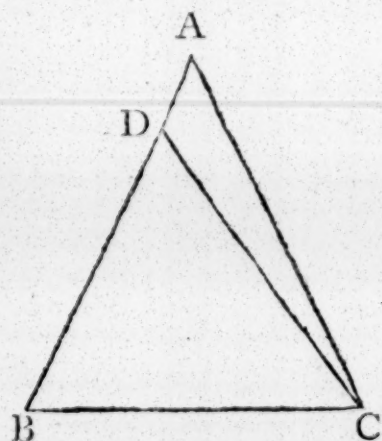
Wherefore the angles at the base of isosceles triangles are equal to one another; and if the equal straight lines be produced, the angles under the base shall be equal to one another. Which was to be demonstrated.

PROP. VI.

If the two angles of a triangle be equal to one another, also the sides extended under the equal angles, shall be equal.

Let there be a triangle the *triangle* ABC, having the angle *contained* by ABC equal to the angle contained by ACB: I say also that a side, the *straight line* AC, is equal to a side the straight line AB.

For if the *straight line* AC is unequal to the *straight line* AB, the one of them is greater: let the straight line AB be the greater; and (by prop. 3.) let the straight line DB be cut off from the greater the *straight line* AB equal to the less the *straight line* AC, and let the *straight line* DC be drawn.



Book I.

Since therefore the *straight line* DB is equal to the *straight line* AC, but the *straight line* BC common; certainly the two DB, BC are equal to the two AC, CB, each to each; and an angle, the *angle contained* by DBC is equal to the *angle contained* by ACB (by *supp.*); wherefore (by *prop. 4.*) the *straight line* DC a base is equal to the *straight line* AB a base, and the triangle ABC will be equal to the triangle DCB, the greater to the less which is absurd (by *com. not. 9.*); wherefore the *straight line* AB is not unequal to the *straight line* AC; therefore equal.

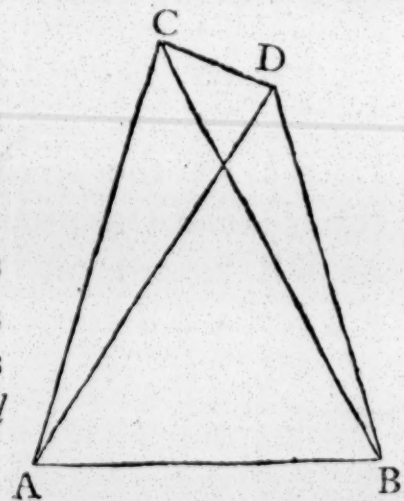
If therefore the two angles of a triangle be equal to one another, also the sides, extended under the equal angles, shall be equal. Which was to be demonstrated.

PROP. VII.

Upon the same straight line, other two straight lines, equal each to each to the same two straight lines, will not join together at different points towards the same parts, if they have the same extremities as the first straight lines.

For if it be possible, upon the same straight line the *straight line* AB, let other two straight lines, the *straight lines* AD, DB equal, each to each, to the same two straight lines, the *straight lines* AC, CB, be joined together at different points, the *points* C and D; the *points* C and D being towards the same parts; having the same extremities the *points* A, B as the first straight lines; so that the one CA be equal to the *straight line* DA, having the same extremity with it, the *point* A; and the other CB to the *right line* DB, having the same extremity with it, the *point* B: and let the straight line CD be drawn.

Since therefore the *straight line* AC is equal to the *straight line* AD, also an angle the *angle contained* by ACD is equal (by *prop. 5.*) to the *angle contained* by ADC: wherefore the *angle contained* by ADC is greater than the *angle contained* by DCB; therefore the *angle contained* by CDB is is greater by much than the *angle contained* by DCB.



Since

Since again the *straight line* CB is equal to the *straight line* DB (by *supp.*) also an angle the *angle contained* by CDB is equal (by *prop. 5.*) to the *angle contained* by DCB: but it has been shewn to be much greater than it, which is impossible. Book I.

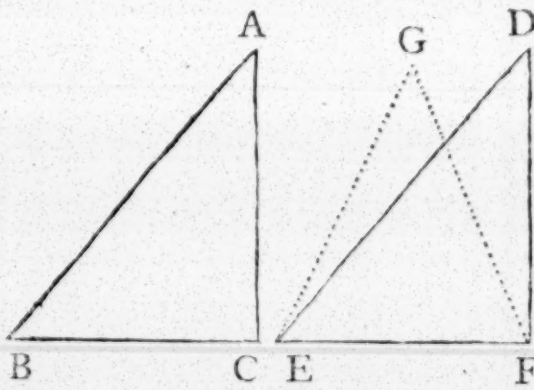
Therefore upon the same *straight line*, other two *straight lines* equal, each to each, to the same two *straight lines* will not join together at different points towards the same parts, if they have the same extremities as the first *straight lines*. Which was to be demonstrated.

P R O P. VIII.

If two triangles have *not only* the two sides equal to the two sides, each to each, but have likewise the base equal to the base; they shall also have the angle equal to the angle, the *angle contained* by the equal *straight lines*.

Let there be two triangles, the *triangles* ABC, DEF having the two sides the *straight lines* AB, AC equal to the two sides the *straight lines* DE, DF, each to each, the one AB to the *straight line* DE, and the other AC to the *straight line* DF; but let them have also a base the *straight line* BC equal to a base the *straight line* EF, I say also that an angle, the *angle contained* by BAC is equal to an angle, the *angle contained* by EDF.

For the triangle ABC being applied to the triangle DEF, and *not only* the point B being put upon the point E, but also the *straight line* BC being put upon the *straight line* EF, the point C also will apply itself to the point F, because the *straight line* BC is equal to the *straight line* EF: certainly the *straight line* BC having applied itself to the *straight line* EF, also the *straight lines* BA, AC will apply themselves to the *straight lines* ED, DF: for if a base as the *straight line* BC should apply itself to a base the *straight line* EF, but the sides BA, AC do not apply themselves to the *straight lines* ED, DF, but change their direction from them as the *straight lines* EG, GF; upon



Book I. the same straight line, other two straight lines equal, each to each, to the same two straight lines will join together at different points towards the same parts, having the same extremities; but (by prop. 7.) they do not join together; and the sides BA, AC, cannot therefore not apply *themselves* to the *straight lines* ED, DF, the base BC applying itself to the base EF, wherefore they do apply themselves, so that also an angle the *angle contained* by BAC will apply *itself* to the angle *contained* by EDF, and shall be equal to it (by com. not. 8.)

If therefore the two triangles have the two sides equal to the two sides, each to each, and have the base equal to the base; they shall also have the angle equal to the angle, that which is contained by the equal straight lines, Which was to be demonstrated.

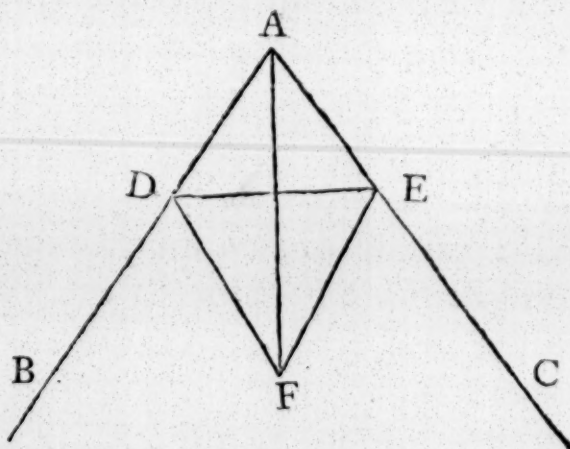
P R O P. IX.

To cut a given rectilineal angle in halves.

Let the given rectilineal angle be, the angle contained by BAC: it is required to cut it in halves.

Let the point D be taken, any accidental *point* in the *straight line* AB; and let the *straight line* AE be cut off (by prop. 3.) from the *straight line* AC equal to the *straight line* AD, and let the straight line DE be drawn (by post. 1.) and (by prop. 1.) let the equilateral triangle the *triangle* DEF be described upon the straight line DE and (by post. 1.) draw the straight line AF: I say that the angle contained by BAC is cut in halves by the straight line AF.

For since the *straight line* AD is equal to the *straight line* AE, but the *straight line* AF common, certainly the two, the *straight lines* DA, AF are equal to the two EA, AF, each to each, and a base the *straight line* DF is equal to a base the *straight line* EF; therefore (by prop. 8.) an angle, the *angle contained* by DAF is equal to an angle, the *angle contained* by EAF.



Wherefore

Wherefore the given rectilineal angle, the *angle contained* by BAC is cut in halves by the straight line AF. Which was to be done. Book I.

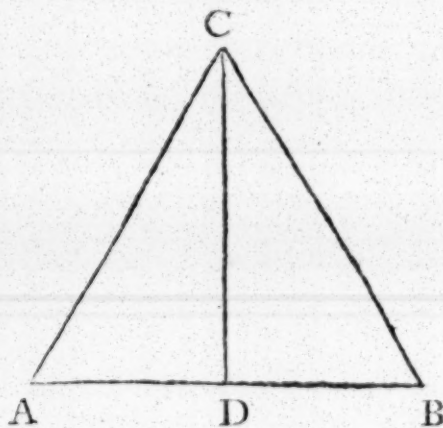
PROP. II.

To cut a given finite straight line in halves.

Let the *straight line* AB be the given finite straight line: it is required to cut the *straight line* AB in halves.

Let an equilateral triangle the *triangle* ABC be described (by prop. 1) upon it, and let the *angle contained* by ACB be cut in halves (by prop. 9.) by the straight line CD: I say that the straight line AB is cut in halves in the point D.

For since the *straight line* AC is equal (by construction) to the *straight line* CB, but the *straight line* CD common, certainly the two AC, CD are equal to the two BC, CD, each to each, and an angle, the *angle contained* by ACD is equal (by const.) to the *angle contained* by BCD: therefore (by prop. 4.) a base the *straight line* AD is equal to a base the *straight line* BD.



Wherefore the given finite straight line the *straight line* AB is cut in halves in the *point* D. Which was to be done.

PROP. XI.

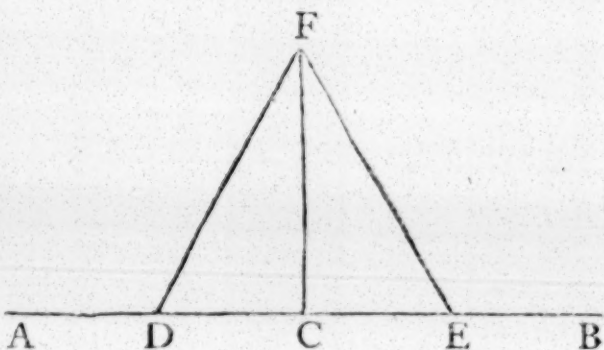
To draw a straight line at right angles to any given straight line from any point given in it.

Let the one, the given straight line be the *straight line* AB, and the other the given point in it the *point* C: it is required from the point C to draw a straight line at right angles to the *straight line* AB.

Let any point which you may accidentally happen upon, as the *point* D be taken in the *straight line* AD; and let the straight line CE be placed (by prop. 3.) equal to CD; and let an equilateral triangle the *triangle* DFE be described upon the *straight line* DE, and let FC be drawn. I say that a straight line the *straight line* FC hath

Book I. hath been drawn at right angles to the given straight line the straight line AB, from any point given in it the point C.

For since the straight line CD is equal to the straight line CE (by const.) and the straight line CF common; certainly the two DC, CF are equal to the two EC, CF each to each; and a base the straight line DF is equal (by const.) to a base the straight line



EF: wherefore (by prop. 8.) an angle the angle contained by DCF is equal to an angle the angle contained by ECF, and they are adjacent or consequent: but when a straight line standing upon a straight line makes the adjacent angles equal to one another; each of the equal angles is a right angle: therefore each of the angles contained by DCF, FCE is a right angle.

Wherefore a straight line the straight line FC hath been drawn at right angles to the given straight line the straight line AB, from any point given in it the point C. Which was to be done.

P R O P. XII.

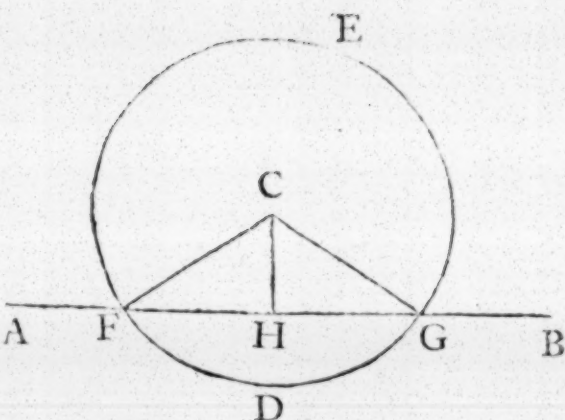
To draw a perpendicular straight line upon any given indefinite straight line, from any given point, which is not in it.

Let the one, the given indefinite straight line, be the straight line AB, and the other, the given point, which is not in it; be the point C; and it is required upon the given indefinite straight line, the straight line AB, from the given point, the point C, which is not in it, to draw a perpendicular straight line.

For on the other side of the straight line AB take any point which you may accidentally happen upon as the point D and with the point C indeed for a center, but at the distance CD let a circle be described, the circle FGE; and let the straight line FG be cut in halves (by prop. 10.) in the point H, and let the straight lines CF, CH, CG be drawn (by post. 1.): I say that upon the given indefinite straight line, the straight line AB, from the given point, the point C, which is not in it, a perpendicular hath been drawn, the straight line CH.

For

For since the straight line FH is equal to the straight line HG (by const.) and the straight line HC common; certainly the two FH, HC are equal to the two GH, HC, each to each; and a base the straight line CF is equal (by def. 15.) to a base the straight line CG: wherefore (by prop. 8.) an angle,



the angle contained by CHF is equal to an angle, the angle contained by GHC, and they are adjacent: but when a straight line standing upon a straight line makes the adjacent angles equal to one another; each of the equal angles is a right angle: and the standing straight line is called a perpendicular to that on which it stands.

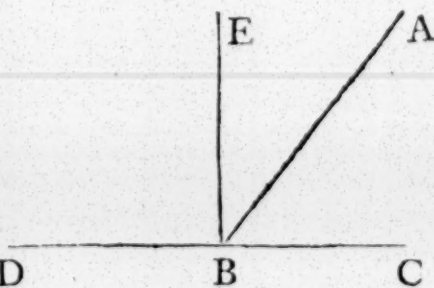
Therefore upon the given indefinite straight line the straight line AB, from the given point, the point C, which is not in it, a perpendicular hath been drawn, the straight line CH. Which was to be done.

P R O P. XIII.

Whenever a straight line standing upon a straight line makes angles: it will make either two right angles, or angles equal to two right angles.

For let any straight line, the straight line AB, standing upon a straight line the straight line CD, make angles, those contained by CBA, ABD: I say that the angles contained by CBA, ABD are either two right angles, or equal to two right angles.

If indeed the angle contained by CBA be equal to the angle contained by ABD, they are (by def. 10.) two right angles: but if not, let the straight line BE be drawn from the point B (by prop. 11.) at right angles to the straight line CD: therefore the angles contained by CBE, EBD are two right angles: and since the



angle contained by CBE is equal to two, the angles contained by CBA, ABE; let the angle contained by EBD, a common one be added:

Book I. added: therefore (by com. not. 2.) the *angles contained* by CBE, EBD are equal to three, the *angles contained* by CBA, ABE, EBD. Again, because the *angle contained* by DBA is equal to two, the *angles contained* by DBE, EBA; let the angle contained by ABC a common one be added: therefore (by com. not. 2.) the angles, the *angles contained* by DBA, ABC are equal to three, the *angles contained* by DBE, EBA, ABC: but the *angles contained* by CBE, EBD have also been proved equal to the same three: but magnitudes which are equal to the same magnitude, are equal to one another: and the *angles contained* by CBE, EBD are therefore equal to the *angles contained* by DBA, ABC: but the *angles contained* by CBE, EBD are two right angles, and the *angles contained* by DBA, ABC are therefore equal to two right angles.

Whenever therefore a straight line standing upon a straight line makes angles; it will make either two right angles, or *angles* equal to two right angles. Which was to be demonstrated.

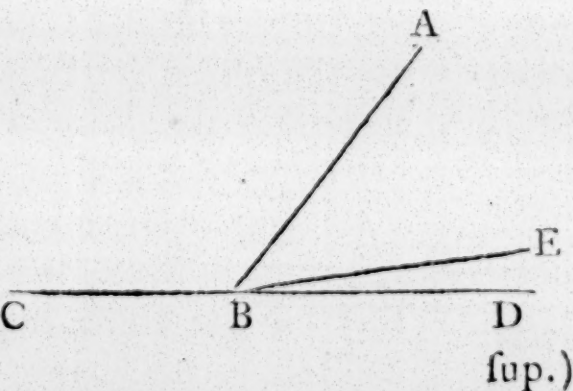
P R O P. XIV.

If, with any straight line, and at the *same* point in it, two straight lines, not lying towards the same parts make the adjacent angles equal to two right angles, the straight lines shall be in a straight line with one another.

For with any straight line, the *straight line* AB, and at the *same* point in it, the *point* B, let two straight lines, the *straight lines* BC, BD not lying towards the same parts, make the adjacent angles, the *angles contained* by ABC, ABD equal to two right angles: I say that the straight line BD is in a straight line with the *straight line* CB.

For if the *straight line* BD is not in a straight line with the *straight line* CB; let the *straight line* BE be in a *straight line* with the straight line CB.

Because therefore a straight line, the *straight line* AB stands upon a straight line, the *straight line* CBE, therefore the angles *contained* by ABC, ABE are equal to two right angles: but the *angles contained* by ABC, ABD are also (by C



sup.) equal to two right angles: therefore the *angles contained* by CBA, ABE are (by com. not. 1.) equal to the *angles contained* by CBA, ABD; let *what is common*, the angle contained by ABC be taken away, therefore what remains the *angle contained* by ABE is equal (by com. not. 3.) to what remains the angle contained by ABD, the less to the greater, which is impossible: therefore the straight line BE is not in a straight line with the *straight line* BC: certainly in the same manner we shall shew that neither any other but the *straight line* BD is: therefore the *straight line* CB is in a straight line with the *straight line* BD.

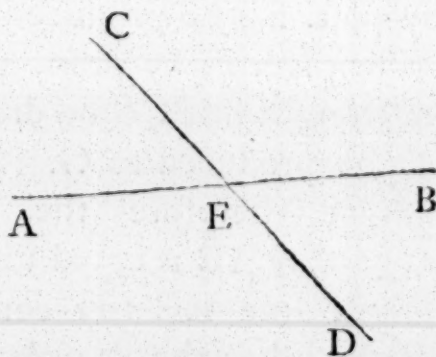
Wherefore if, with any straight line, and at the *same point* in it, two straight lines, not lying towards the same parts, make the adjacent angles equal to two right angles, the straight lines shall be in a straight line with one another. Which was to be demonstrated.

P R O P. XV.

If two straight lines cut one another they shall make the angles at the vertex equal to each other.

For let two straight lines, the *straight lines* AB, CD cut one another at the point E: I say that the one the *angle contained* by AEC is equal to the angle contained by DEB and the other *contained* by CEB to the *angle contained* by AED.

For since a straight line, the *straight line* AE stands upon a straight line the *straight line* CD, making angles, the *angles contained* by CEA, AED: therefore the angles *contained* by CEA, AED are equal to two right angles (by prop. 13.): again, because a straight line, the *straight line* DE stands upon a straight line, the *straight line* AB, making angles the *angles contained* by AED, DEB: therefore the *angles contained* by AED, DEB are equal to two right angles: but also the *angles contained* by CEA, AED have been demonstrated to be equal to two right angles: therefore (by com. not. 1.) the *angles contained* by CEA, AED are equal to the *angles contained* by AED, DEB: let the *angle contained* by AED



Book I. *which* is common be taken away, wherefore (by com. not. 3.) the angle contained by CEA which remains is equal to the angle contained by BED which remains: certainly in the same manner it will be shewn, that also the angles contained by CEB, DEA are equal.

If therefore two straight lines cut one another they shall make the angles, at the vertex equal to each other. Which was to be demonstrated.

Corollary. Certainly it is manifest from this, that also whatever number of straight lines cut one another, they will make the angles at the section equal to four right angles.

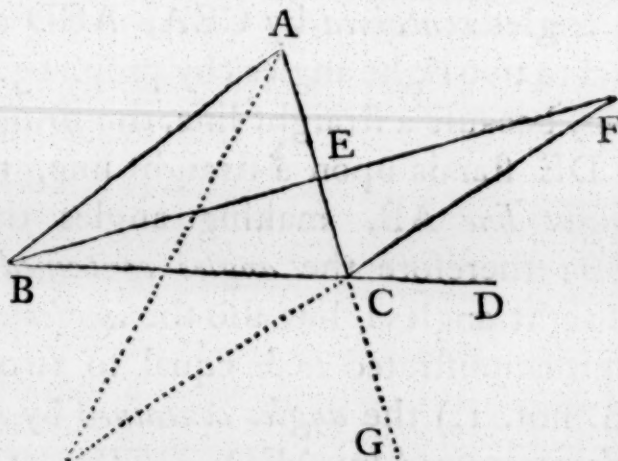
PROP. XVI.

One of the sides of any triangle being produced, the outward angle is greater than either of those *two which are* within and opposite.

Let there be a triangle, the *triangle* ABC, and let one side of it the straight line BC be produced to the point D: I say that the angle without, the angle contained by ACD is greater than either of those *two which are* within and opposite, the angles contained under CBA, BAC.

Let the straight line AC be cut in halves in the point E (by prop. 10.); and the straight line BE being drawn (by post. 1.) let it (by post. 2.) be produced to the point F, and let the straight line EF be placed (by prop. 3.) equal to the *straight line* BE and let the straight line CF be drawn (by post. 1.), and let the straight line AC be produced to G.

Since therefore the one *straight line* AE is equal to the straight line EC (by const.) and the other the *straight line* BE to the *straight line* EF; certainly the two AE, EB are equal to the two CE, EF, each to each, and an angle, the angle contained by AEB, is equal to an angle, the angle



contained

contained by FEC (by prop. 15.) for *they are* vertical : wherefore Book I.
 (by prop. 4.) a base the *straight line* AB is equal to a base the
 straight line CF and the triangle ABE is equal to the triangle
 FEC, and the remaining angles are equal to the remaining angles,
 under which the equal sides are extended, each to each : wherefore
 the angle contained by BAE is equal to the angle contained by ECF :
 but the angle contained by ECD is greater (by com. not. 9.) than
 the angle contained by ECF : wherefore the angle contained by
 ACD is greater than the angle contained by BAE : in like manner
 also the straight line BC having been cut in halves, the angle con-
 tained by BCG, that is (by prop. 15.) the angle contained by
 ACD, is also greater than the angle contained by ABC.

Wherefore one of the sides of any triangle being produced, the
 outward angle is greater than either of those *two which are* within
 and opposite. Which was to be demonstrated.

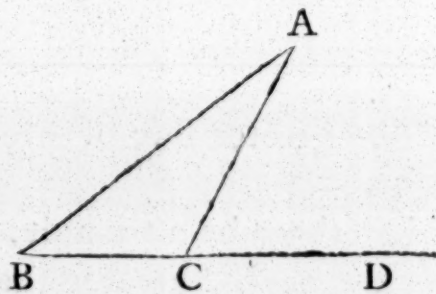
PROP. XVII.

The two angles of any triangle are less than two right angles,
 being interchanged every way.

Let there be a triangle the *triangle* ABC ; I say that the two an-
 gles of the triangle ABC are less than two right angles, being in-
 terchanged every way.

For let the *straight line* BC be produ-
 ced to the point D.

And since the angle contained by
 ACD is an outward angle of the triangle
 ABC, it is greater than the inward and
 opposite, the angle contained by ABC :
 let a common one be added, the angle
 contained by ACB : therefore the angles contained by ACD, ACB
 are greater (by com. not. 4.) than those contained by ABC, BCA ;
 but the angles contained by ACD, ACB are equal to two right an-
 gles ; therefore those contained ABC, BCA are less than two right
 angles. And in like manner we shall shew that the angles contained
 by BAC, ACB are less than two right angles, as also those contained
 by CAB, ABC.



Book I.

Wherefore the two angles of any triangle are less than two right angles, being interchanged every way.

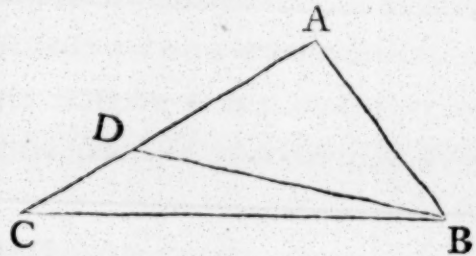
PROP. XVIII.

The greater side of every triangle subtends the greater angle.

Let there be a triangle, the *triangle* ABC, having the side AC greater than the side AB, I say that also the angle contained by ABC is greater than the angle contained by BCA.

For because the *straight line* AC is greater than the *straight line* AB let the *straight line* AD be placed (by prop. 3.) equal to the *straight line* AB, and let the *straight line* BD be drawn.

And since the angle contained by ADB is an outward angle of a triangle, the *triangle* BDC, it is greater (by prop. 16.) than the inward and opposite, the angle contained by DCB. But the angle contained by ADB is equal to the angle contained by ABD (by prop. 5.), because the side AB is equal to the side AD: therefore also the angle contained by ABD is greater than the angle contained by ACB; wherefore the angle contained by ABC is much greater than the angle contained by ACB.



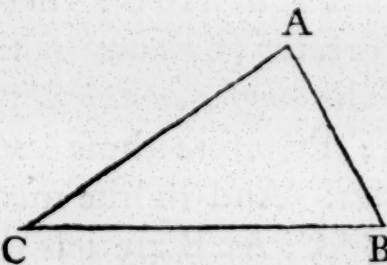
Therefore the greater side, of every triangle, subtends the greater angle. Which was to be demonstrated.

PROP. XIX.

The greater side of every triangle is extended under the greater angle.

Let there be a triangle, the *triangle* ABC, having the angle contained by ABC greater than the angle contained by BCA; I say that also the side AC is greater than the side AB.

For if not: certainly the *straight line* AC is equal to the *straight line* AB, or less: but indeed the *straight line* AC is not equal to the *straight line* AB, for then also an angle, the angle contained by ABC would be equal (by prop. 5.) to the angle contained by ACB: but (by supp.) it is not: therefore the *straight line*



line AC is not equal to the straight line AB; neither is the straight line AC less than the straight line AB; for if it were less, then (by prop. 18.) an angle the angle contained by ABC would be less than the angle contained by ACB: but (by supp.) it is not: therefore the straight line AC is not less than the straight line AB; but it has been demonstrated, that it is not equal to it; wherefore the side AC is greater than the side AB. Book I.

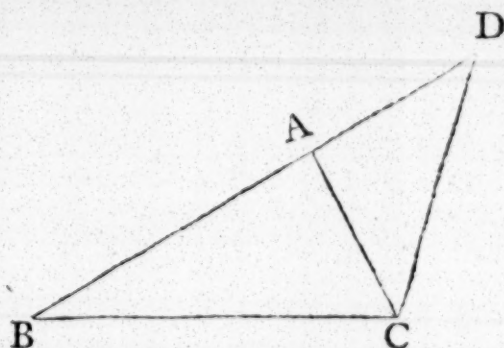
Wherefore the greater side of every triangle is extended under the greater angle. Which was to be demonstrated.

P R O P. XX.

The two sides of every triangle are greater than the remaining side, being interchanged every way.

For let there be a triangle, the triangle ABC: I say that the two sides of the triangle ABC are greater than the remaining side, being interchanged every way: viz. BA, AC greater than BC; and AB, BC greater than AC; lastly BC, CA greater than AB.

For let the straight line BA be produced to the point D, and let the straight line DA be placed (by prop. 3.) equal to CA, and let the straight line DC be drawn.



Since therefore the straight line DA is equal to the straight line AC, also (by prop. 5.) the angle contained by ADC is equal to the angle contained by ACD: but the angle contained by BCD is greater than the angle contained by ACD (by com. not. 9.): wherefore the angle contained by BCD is also greater than the angle contained by ADC: and because the triangle DCB is a triangle having the angle contained by BCD greater than the angle contained by BDC, and (by prop. 19.) the greater side is extended under the greater angle, wherefore the straight line DB is greater than the straight line BC; but the straight line BD is equal to the straight lines AB, AC: therefore the straight lines BA, AC are greater than the straight line BC: certainly in the same manner we shall demonstrate that also AB, BC are greater than AC; and BC, CA greater than AB. There-

Book I.

Therefore of every triangle the two sides are greater than the remaining *side*, being interchanged every way. Which was to be demonstrated.

PROP. XXI.

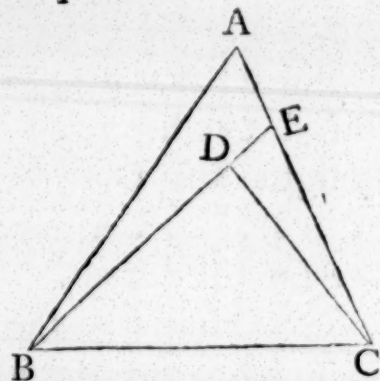
If two straight lines be joined together within a triangle, upon one of its sides from the extremities of it, the joined lines shall indeed be less than the two remaining sides of the triangle, but will contain a greater angle.

For let two straight lines, the *straight lines* BD, DC be joined together within a triangle, the *triangle* ABC upon one of the sides the *straight line* BC drawn from the extremities the *points* B, C; I say that the *straight lines* BD, DC are indeed less than the two remaining sides of the triangle the *straight lines* BA, AC; but contain a greater angle, the *angle contained* by BDC greater than the *angle contained* by BAC.

For let the *straight line* BD be produced to the *point* E.

And because the two sides of any triangle are greater than the remaining *side*, therefore the two sides of the triangle ABE, the *straight lines* AB, AE are greater than the *straight line* BE (by prop. 20.); let the *straight line* EC a common *one* be added: therefore (by com. not. 4.) the *straight lines* BA, AC are greater than the *straight lines* BE, EC. Again, because the two sides of the triangle CED, the *straight lines* CE, ED are greater than the *straight line* CD, let the *straight line* DB a common *one* be added; therefore (by com. not. 4.) the *straight lines* CE, ED are greater than the *straight lines* CD, DB: but the *straight lines* BA, AC have been shewn to be greater than the *straight lines* BE, EC; therefore BA, AC are greater by much than BD, DC.

Again, because the outward angle of every triangle is greater than the inward and opposite (by prop. 16.): therefore the outward angle of the triangle CDE, the *angle contained* by BDC is greater than that *contained* by CED: For the same reason therefore also the outward angle of the triangle ABE, the *angle contained*



tained by CEB is greater than that contained by BAC : but the *angle contained* by BDC has been demonstrated to be greater than that contained by CEB : therefore the *angle contained* by BDC is greater by much than that contained by BAC. Book I.

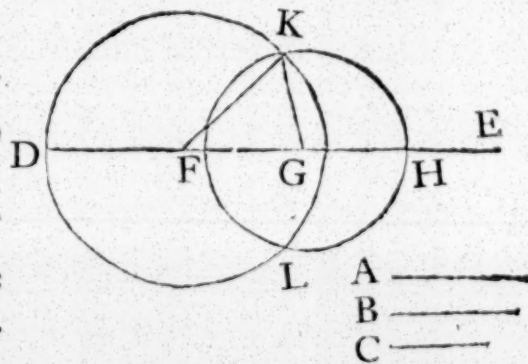
If therefore two straight lines be joined together within a triangle, upon one of its sides from the extremities of it, the joined lines shall indeed be less than the two remaining sides of the triangle, but will contain a greater angle. Which was to be demonstrated.

P R O P. XXII.

To describe a triangle of three straight lines, which are equal to three given straight lines : but it is necessary that two of them be greater than the remaining *one* ; being interchanged every way.

Let the three given straight lines be A, B, C ; of which let the two be greater than the remaining *one*, being interchanged every way, that is A and B greater than C : A and C greater than B ; as also B and C greater than A : it required to describe a triangle of *straight lines* equal to A, B, C.

Draw any straight line DE terminated at the point D but indefinite towards the point E ; and (by prop. 3.) let DF be made equal to A, and FG equal to B ; but GH equal to C : and with the point F for a center, and at the distance FD let the circle DKL be described ; and again with the point G for



a center, and at the distance GH let the circle KLH be described ; and draw KF, KG : I say that the triangle KFG hath been described of three straight lines equal to A, B, C.

For because the point F is the center of the circle DKL, FD is equal to FK : but FD is equal (by const.) to A, therefore FK is equal to A. Again because the point G is the center of the circle LKH, (by def. 15.) GK is equal to GH ; but GH is equal to C, therefore GK is equal to C : but FG is also equal to B ; therefore the three straight lines KF, FG, GK are equal to the three A, B, C. There-

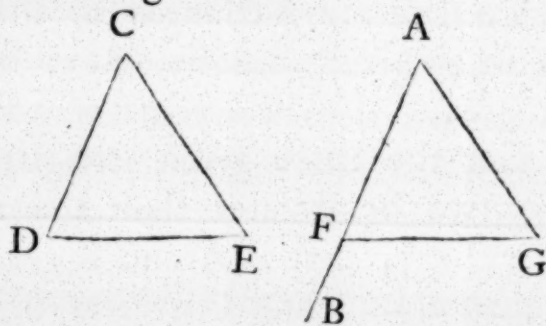
Book I. Therefore the triangle KFC is described of three straight lines, which are equal to the three given straight lines A, B, C. Which was to be done.

P R O P. XXIII.

With a given straight line, and at a point in it, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and the *point* A the point in it; and DCE the given rectilineal angle: but it is required with the given straight line AB and at the point A in it to make a rectilineal angle equal to the given rectilineal angle DCE.

Take in each of the *lines* CD, CE any points whatever D and E, and join DE, and (by prop. 22.) describe the triangle AGF of three straight lines, which are equal to the three, CD, DE, CE; so that CD be equal to AF; CE to AG, and also DE to FG.



Since therefore the two DC, CE are equal to the two FA, AG, each to each, and the base DE equal to the base FG; therefore (by prop. 8.) the angle DCE is equal to the angle FAG.

Wherefore with a given straight line AB, and at a point in it the *point* A, a rectilineal angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

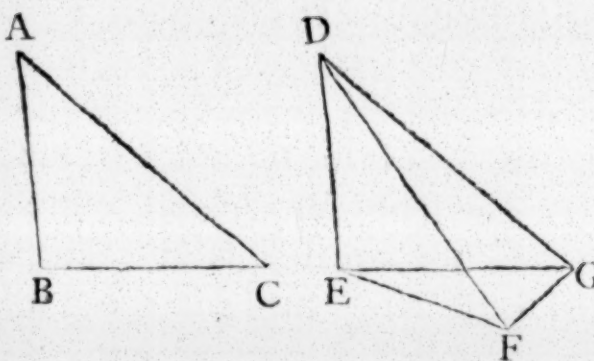
P R O P. XXIV.

If two triangles have the two sides equal to the two sides, each to each, but have the angle greater than the angle, the *angle* contained by the equal straight lines: also they will have the base greater than the base.

Let there be two triangles the *triangles* ABC, DEF having the two sides AB, AC equal to the two sides DE, DF, each to each; AB to DE, and AC to DF; but let an angle the *angle* contained by BAC be greater than the angle *contained* by EDF; I say that the base BC is greater than the base EF.

For

For because the angle BAC is greater than the angle EDF; let there be made, with the straight line DE and at the point D in it, the angle EDG equal to the angle BAC; and let DG be made equal (by prop. 3.) to either of the *lines* AC, DF; and let GE, GF be joined.



Since therefore AB is equal to DE and AB to DG; certainly the two BA, AC are equal to the two ED, DG, each to each; and the angle BAC is equal (by const.) to EDG therefore the base BC is equal to the base EG. Again because DG is equal to DF, the angle DFG is equal (by prop. 5.) to the angle DGF; therefore the *angle* DFG is greater than the *angle* EGF; therefore the *angle* EFG is greater by much than the *angle* EGF; and because there is a triangle, the *triangle* EFG, having the angle EFG greater than the *angle* EGF; but (by prop. 19.) the greater side is extended under the greater angle: therefore the side EG is greater than EF: and EG is equal to BC (by part. 1. of this prop.); wherefore also BC is greater than EF.

Wherefore if two triangles have the two sides equal to the two sides, each to each, and have the angle greater than the angle, the *angle contained* by the equal straight lines; they will also have the base greater than the base. Which was to be demonstrated.

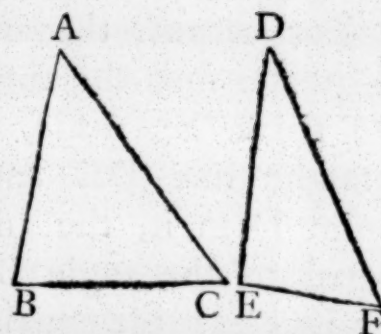
P R O P. XXV.

If two triangles have the two sides equal to the two sides each to each, and have the base greater than the base; they will also have the angle greater than the angle, the *angle* contained by the equal straight lines.

Let there be two triangles the *triangles* ABC, DEF having the two sides BA, AC equal to the two sides DE, DF, each to each, AB to DE and AC to DF; but let the base BC be greater than the base EF: I say also that an angle, the *angle* BAC is greater than an angle, the *angle* EDF.

Book I.

For if not; certainly it is equal to it, or less. But certainly the angle BAC is not equal to EDF; for then (by prop. 4.) the base BC would be equal to the base EF: but it is not (by sup.); therefore the angle BAC is not equal to EDF: but neither indeed is it less; for if it were less, also the base BC *would be less* (by prop. 24.) than the base EF: but it is not (by sup.); therefore the angle BAC is not less than EDF: but it has been demonstrated that neither is it equal: wherefore the angle BAC is greater than EDF.



Wherefore if two triangles have the two sides equal to the two sides each to each, and have the base greater than the base; they will also have the angle greater than the angle, the *angle* contained by the equal straight lines. Which was to be demonstrated.

P R O P. XXVI.

If two triangles have the two angles equal to the two angles, each to each, and one side equal to one side; either the side which is at the equal angles, or that which is extended under one of the equal angles: they will also have the *two* remaining sides equal to the two remaining sides each to each, and the remaining angle *equal* to the remaining angle.

Let there be two triangles the *triangles* ABC, DEF having the two angles ABC, BCA equal to the two *angles* DEF, EFD each to each; the *angle* ABC equal to the *angle* DEF; and the *angle* BCA *equal* to EFD: and let them have also one side equal to one side:

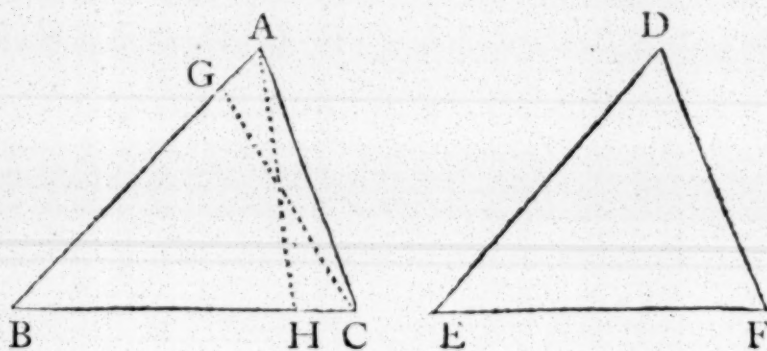
First, the side at the equal angles, the *straight line* BC equal to EF: I say also that they will have the *two* remaining sides equal to the *two* remaining sides, each to each; AB *equal* to DE and AC to DF; and the remaining angle *equal* to the remaining angle, the *angle* BAC *equal* to EDF.

For if AB is unequal to DE, one of them will be greater *than the other*: let AB be the greater and make GB equal (by prop. 3.) to DE, and let CG be joined.

Where-

Wherefore because BG is equal to DE (by const.) and BC to EF (by supp.) certainly the two BG, BC are equal to the two DE, EF each to each; and the angle GBC is equal (by supp.) to DEF: therefore (by prop. 4.) the base GC is equal to the base DF; and the triangle GCB will be equal to the triangle DEF; and the remaining angles will be equal to the remaining angles, each to each, under which the equal sides are extended: therefore the angle GCB is equal to the angle DFE; but the angle DFE is supposed equal to BCA; therefore BCG is equal to BCA the less to the greater, which is impossible; therefore AB is not unequal to DE; therefore equal: but BC is also equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each, and the angle ABC is equal to the angle DEF; wherefore the base AC is equal to the base DF and the remaining angle BAC is equal to the remaining angle EDF.

But again, let the sides extended under the equal angles be equal, as AB equal to DE: I say again that also the remaining sides will be equal to the remaining sides; AC to



DF and BC to EF; and besides the remaining angle BAC is equal to the remaining angle EDF.

For if BC be unequal to EF, one of them is greater *than the other*; let BC, if it be possible, be the greater, and make (by prop. 3.) BH equal to EF, and let AH be joined.

And since BH is equal to EF (by const.) and AB to DE (by supp.); certainly the two AB, BH is equal to the two DE, EF, each to each, and they contain equal angles; therefore (by prop. 4.) the base AH is equal to the base DF, and the triangle ABH is equal to the triangle DEF, and the remaining angles will be equal to the remaining angles, each to each, under which the equal sides are extended; wherefore the angle BHA is equal to EFD; but the angle EFD is equal (by supp.) to the angle BCA; and (by com. not. 1.) BHA is therefore equal to BCA; the outward angle of the triangle AHC viz. BHA equal to the inward and opposite

D 2

BCA

Book I. BCA, which (by prop. 16.) is impossible: wherefore BC is not unequal to EF, therefore *it is* equal: but AB is also equal to DE; therefore the two AB, BC are equal to the two DE, EF and they contain equal angles; therefore (by prop. 4.) the base AC is equal to the base DF, and the triangle ABC is equal to the triangle DEF and the remaining angle BAC is equal to the remaining angle EDF.

If therefore two triangles have the two angles equal to the two angles, each to each, and have one side equal to one side; either the side which is at the equal angles, or that which is extended under one of the equal angles: they will also have the *two* remaining sides equal to the *two* remaining sides, each to each, and the remaining angle *equal* to the remaining angle. Which was to be demonstrated.

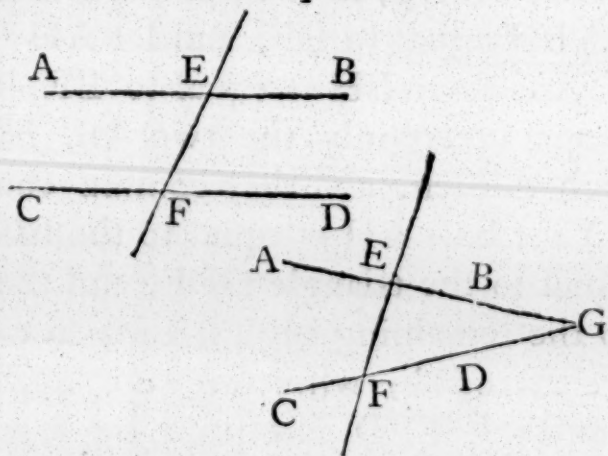
P R O P. XXVII.

If a straight line falling upon two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to one another.

For let the straight line EF falling upon the two straight lines AB, CD make the alternate angles AEF, EFD equal to one another. I say that the *straight line* AB is parallel to the straight line CD.

For if not, AB and CD being produced will meet, either towards the the parts B, D or towards A, C: let them be produced and let them meet towards the parts B, D in the *point* G.

Certainly (by prop. 16.) the outward angle AEF of the triangle EFG is greater than the inward and opposite angle EFG; but it is also (by supp.) equal, which is impossible: therefore AB, CD being produced, will not meet towards the parts B, D; certainly in the same manner it will be



shewn, that neither *will they meet* towards A, C; but those *straight lines*

lines meeting one another towards neither parts are parallel: there- Book 1.
fore AB is parallel to CD.

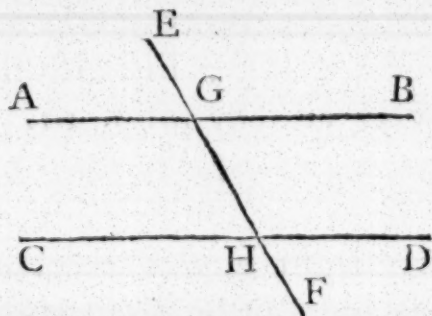
Wherefore if a straight line falling upon two straight lines make the alternate angles equal to one another, the straight lines will be parallel. Which was to be demonstrated.

P R O P. XXVIII.

If a straight line falling upon two straight lines make the outward angle equal to the inward and opposite, and towards the same parts; or make the inward angles and towards the same parts equal to two right angles: the straight lines will be parallel to one another.

For let the straight line EF falling upon the two straight lines AB, CD make the outward angle EGB equal to the inward and opposite and towards the same parts the angle GHD; or the inward and towards the same parts, the *angles* BGH, GHD equal to two right angles; I say that AB is parallel to CD.

For because the *angle* EGB is equal to the *angle* GHD; but EGB is equal (by prop. 15.) to AGH, also the *angle* AGH is therefore (by com. not. 1.) equal to GHD: and they are alternate; therefore (by prop. 27.) AB is parallel to CD.



Again because the angles BGH, GHD are (by supp.) equal to two right angles; and also the *angles* AGH, BGH are equal (by prop. 13.) to two right angles; therefore (by com. not. 1.) the *angles* AGH, BGH are equal to the *angles* BGH, GHD: let the common *angle* BGH be taken away, therefore (by com. not. 3.) the remaining *angle* AGH is equal to the remaining *angle* GHD: and they are alternate; wherefore (by prop. 27.) AB is parallel to CD.

Wherefore if a straight line falling upon two straight lines make the outward angle equal to the inward and opposite and towards the same parts; or make the inward angles and towards the same parts
equal

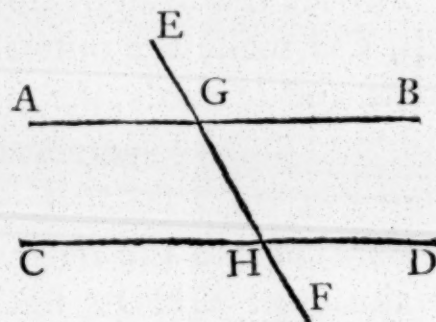
Book I. equal to two right angles : the straight lines will be parallel to one another. Which was to be demonstrated.

P R O P. XXIX.

Or a straight line falling upon the two parallel straight lines makes the alternate angles equal to one another ; and the outward angle equal to the inward and opposite and towards the same parts ; and the inward angles and towards the same parts equal to two right angles.

For let the straight line EF fall upon parallel straight lines, the straight lines AB, CD : I say that it makes the alternate angles AGH, GHD equal ; and the outward angle EGB equal to GHD the inward and opposite and towards the same parts ; and BGH, GHD the inward and towards the same parts equal to two right angles.

For if AGH be unequal to GHD, one of them is greater : let AGH be the greater : and because AGH is greater than GHD, let BGH *which is common* be added ; therefore (by com. not. 4.) the angles AGH, BGH are greater than BGH, GHD : But also the angles AGH, BGH are (by prop. 13.) equal to two right angles ; and therefore the angles BGH, GHD are less than two right angles ; but those straight lines which are produced indefinitely from angles less than two right angles meet one another (by com. not. 11.) ; therefore AB and CD being produced indefinitely will meet one another ; but they do not meet, because they are supposed parallel : therefore AGH is not unequal to GHD, therefore it is equal.



But the angle AGH is equal to the angle EGB (by prop. 15.) ; and therefore (by com. not. 1.) EGB is equal to GHD.

Let the common angle BGH be added : wherefore the angles EGB, BGH are equal to the angles BGH, GHD : but the angles EGB, BGH are (by prop. 13.) equal to two right angles ; and therefore the angles BGH, GHD are (by com. not. 1.) equal to two right angles.

Or a straight line, therefore, falling upon the two parallel straight lines, makes the alternate angles equal to one another ; and the

the outward *angle* equal to the inward and opposite and towards the same parts; and the inward angles and towards the same parts equal to two right angles. Which was to be demonstrated. Book I.

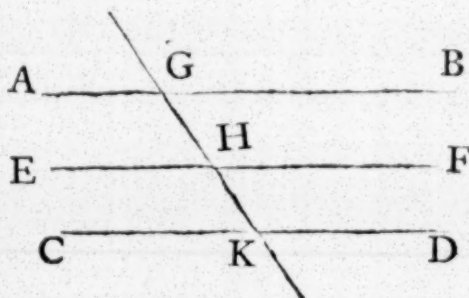
P R O P. XXX.

Straight lines parallel to the same straight line are also parallel to one another.

Let each of the *straight lines* AB, CD be parallel to EF: I say also that AB is parallel to CD.

For let the straight line GK fall upon them.

And because the straight line GK hath fallen upon parallel straight lines AB, EF therefore (by prop. 29.) the angle AGH is equal to GHF: again, because the straight line GK hath fallen upon the parallel straight lines EF, CD the angle GHF is equal to GKD: but the angle AGK hath been proved equal to GHF: and therefore (by com. not. 1.) AGK is equal to GKD; and they are alternate *angles*: therefore AB is parallel to CD (by prop. 27.)



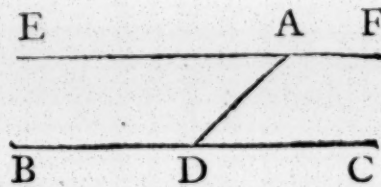
Wherefore the *straight lines* which are parallel to the same straight line, are parallel to one another. Which was to be demonstrated.

P R O P. XXXI.

Through a given point to draw a straight line parallel to a given straight line

Let the given point be the *point* A; and the given straight line the *straight line* BC; it is required through the point A to draw a straight line parallel to the straight line BC.

Let any point whatever the *point* D be taken in BC; and let AD be joined. And let the *angle* DAE be made, with the straight line AD and at the point A in it, equal to the angle ADC (by prop. 23.): and let the straight line AF be produced in a straight line with AE.



And

Book I. And because the straight line AD falling upon the two straight lines BC, EF hath made the alternate angles, the angles EAD, ADC equal to one another, therefore (by prop. 27.) EF is parallel to BC.

Wherefore through a given point the point A a straight line EAF hath been drawn parallel to the given straight line BC. Which was to be done.

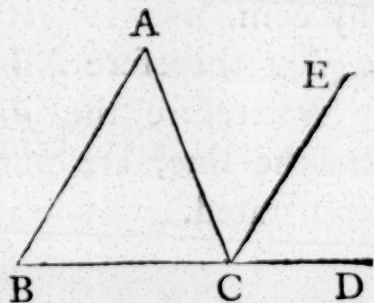
P R O P. XXXII.

One of the sides of any triangle being produced, the outward angle is equal to the two inward and opposite angles: and the three inward angles of the triangle are equal to two right angles.

Let there be a triangle, the triangle ABC, and let one of its sides BC be produced to the point D: I say that the outward angle ACD is equal to the two inward and opposite ones CAB, ABC; and that the three inward angles of the triangle viz. ABC, BCA, CAB are equal to two right angles.

For let CE be drawn (by prop. 31.) through the point C parallel to the straight line AB.

And because AB is parallel to CE and AC hath fallen upon them; the alternate angles BAC, ACE are equal (by prop. 27.): Again, because AB is parallel to CE and the straight line BD hath fallen upon them; the outward angle ECD is equal to the inward and opposite ABC (by prop. 29.): but ACE has been also shewn to be equal to BAC; the whole therefore, the outward angle ACD is equal to the two inward and opposite ones BAC, ABC.



Let the common angle ACB be added: therefore the angles ACD, ACB are equal to the three angles ABC, BAC, BCA; but ACD, ACB are (by prop. 13.) equal to two right angles; and therefore (by com. not. 1.) ACB, CBA, CAB are equal to two right angles.

Wherefore one of the sides of any triangle being produced, the outward angle is equal to the two inward and opposite angles: and the three inward angles of the triangle are equal to two right angles. Which was to be demonstrated.

P R O P.

P R O P. XXXIII.

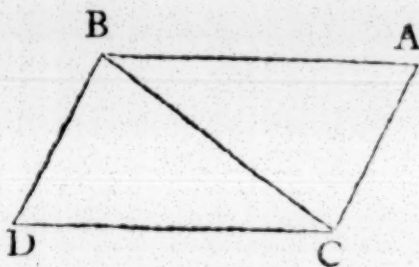
Book I.

The straight lines joining equal and parallel *straight lines* towards the same parts, are also themselves equal and parallel.

Let the *straight lines* AB, CD be equal and parallel; and let the straight lines AC, BD join them towards the same parts: I say that AC, BD are also equal and parallel.

For let BC be joined.

And because AB is parallel to CD, and BC hath fallen upon them; the alternate angles, the *angles* ABC, BCD are (by prop. 29.) equal to one another: and because (by supp.) AB is equal to CD, and the *straight line* BC common; certainly the two AB, BC are equal to the two DC, CB; and the angle ABC is equal to BCD; therefore (by prop. 4.) the base AC is equal to the base BD; and the triangle ABC is equal to the triangle BCD; and the remaining angles will be equal to the remaining angles, under which the equal sides are extended, each to each; therefore the angle ACB is equal to the *angle* CBD: and because the straight line BC falling upon the two straight lines AC, BD hath made the alternate angles ACB, CBD equal to one another; therefore (by prop. 27.) AC is parallel to BD; and it has been demonstrated also *to be* equal.



Wherefore the straight lines joining equal and parallel *straight lines* towards the same parts are also themselves equal and parallel. Which was to be demonstrated.

P R O P. XXXIV.

The opposite sides and also the *opposite* angles of parallelogram spaces are equal to one another, and the diameter cuts the spaces in halves.

Let there be a parallelogram, the *parallelogram* ACDB and its diameter the *straight line* BC: I say that the opposite sides and also the angles of the parallelogram ACDB are equal to one another, and the diameter BC cuts it in halves.

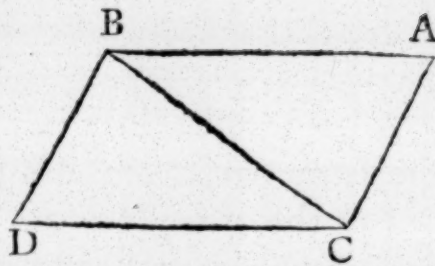
VOL. I.

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For

Book I.

For because AB is parallel to CD and the straight line BC hath fallen upon them; the alternate angles ABC, BCD are equal to one another (by prop. 29.): Again because AC is parallel to BD and BC hath fallen upon them; the alternate



angles ACB, CBD are (by prop. 29.) equal to one another: There are two triangles ABC, CBD having the two angles ABC, BCA equal to the two angles BCD, CBD, each to each; and one side equal to one side, the *side* at the equal angles, the *straight line* BC common to them both: therefore they will have the remaining sides equal to the remaining sides, each to each, (by prop. 26.) and the remaining angle *equal* to the remaining angle; wherefore the side AB is equal to CD; and AC to BD; and the angle BAC equal to the angle BDC; and since the angle ABC is equal to the angle BCD; and the angle CBD, to the angle ACB; therefore the whole (by com. not. 2.) angle ACD is equal to the whole ABD; but the angle BAC hath been proved equal to the angle BDC.

Wherefore the opposite sides and also the angles of parallelogram spaces are equal to one another.

But I say also that the diameter cuts them in halves: for because AB is equal to CD; and BC common; certainly the two AB, BC are equal to the two DC, CB; each to each; and the angle ABC is equal to the angle BCD; therefore (by prop. 4.) also the base AC is equal to the base BD; and the triangle ABC is equal to the triangle BCD.

Wherefore the diameter BC cuts in halves the parallelogram ACDB. Which was to be demonstrated.

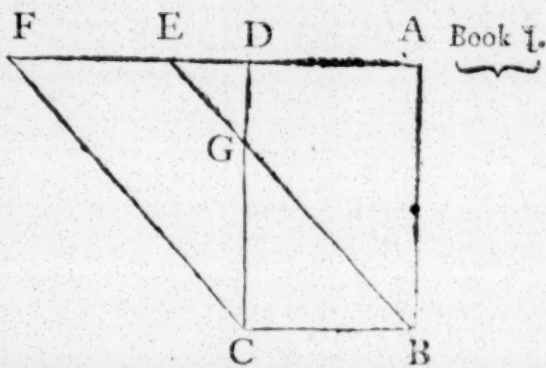
PROP. XXXV.

The parallelograms are equal to one another, which are upon the same base and between the same parallel lines.

Let there be parallelograms, the *parallelograms* ABCD, EBCF being upon the same base BC and between the same parallel lines AF and BC: I say that the *parallelogram* ABCD is equal to the *parallelogram* EBCF.

For

For because ABCD is a parallelogram, AD is (by prop. 34.) equal to BC : and for the same reason EF is equal to BC ; so that (by com. not. 1.) AD is equal to EF ; and DE is common ; therefore (by com. not. 2.) the whole AE is equal to the whole DF ; but (by prop. 34.) AB is also equal to DC ; therefore the two EA, AB are equal to the two FD, DC, each to each ; and the angle FDC is equal to the angle EAB (by prop. 29.) the outward to the inward : therefore the base EB is equal to the base FC ; and the triangle EAB is equal to the triangle FCD : let the common *triangle* DEG be taken away ; therefore the remainder, the trapezium ABGD is equal to the remainder the trapezium EGCF ; let a common part be added, the triangle GBC : therefore (by com. not. 2.) the whole, the parallelogram ABCD is equal to the whole, the parallelogram EBCF.



Therefore the parallelograms, being upon the same base, and between the same parallels are equal to one another. Which was to be demonstrated.

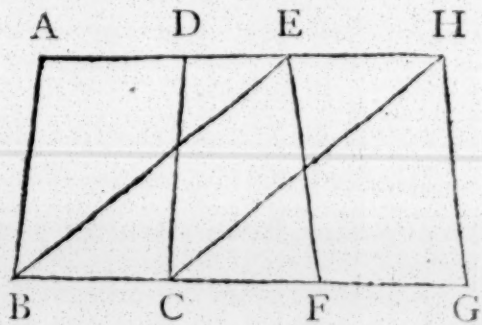
P R O P. XXXVI.

The parallelograms, being upon equal bases and between the same parallels, are equal to one another.

Let there be parallelograms, the *parallelograms* ABCD, EFGH upon equal bases the *straight lines* BC, FG ; and between the same parallels AH, BG : I say that the parallelogram ABCD is equal to the *parallelogram* EFGH.

For let BE, CH be joined.

And because BC is (by supp.) equal to FG, and also FG (by prop. 34.) to EH ; therefore (by com. not. 1.) BC is equal to EH ; but they are likewise (by sup.) parallel ; and BE and CH join them ; but *straight lines* which join equal and parallel straight lines towards the same parts are equal and parallel (by prop. 33.) ; therefore EB, CH are equal and parallel ; wherefore EBCH is a parallelogram, and is equal to



E 2

ABCD

Book I. $\underbrace{\hspace{1em}}$ ABCD (by prop. 35.) for it has the same base with it the *straight line* BC, and it is between the same parallels with it the *straight lines* BC, AH. Certainly for the same reason also EFGH is equal to EBCH; therefore (by com. not. 1.) the parallelogram ABCD is equal to EFGH.

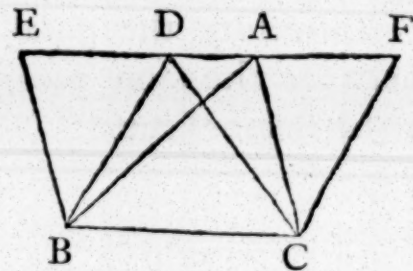
Wherefore the parallelograms, being upon equal bases and between the same parallels are equal to one another. Which was to be demonstrated.

PROP. XXXVII.

The triangles, which are upon equal bases and between the same parallels, are equal to one another.

Let there be triangles, the *triangles* ABC, DBC, being upon the same base the *straight line* BC; and between the same parallels the *straight lines* AD, BC: I say that the triangle ABC is equal to the triangle DBC.

Let AD be produced towards both parts to the points E and F; and through the point B let BE be drawn parallel to AC; and through the point C let CF be drawn parallel to BD.



Therefore each of the *figures* EBCA, DBCF is (by const. and sup.) a parallelogram: and EBCA is equal (by prop. 35.) to DBCF for they are upon the same base BC and between the same parallels BC, EF: and the triangle ABC is half of the parallelogram EBCA (by prop. 34.), for the diameter AB cuts it in halves; but the triangle DBC is half of the parallelogram DBCF, for the diameter DC cuts it in halves: but the halves of equal magnitudes are equal; wherefore the triangle ABC is equal to the triangle DBC.

Therefore the triangles, which are upon the same base, and between the same parallels, are equal to one another. Which was to be demonstrated.

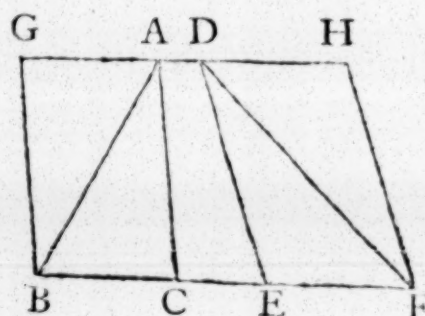
PROP. XXXVIII.

The triangles, which are upon equal bases and between the same parallels, are equal to one another.

Let

Let ABC , DEF be triangles which are upon equal bases BC , EF ; and between the same parallels the *straight lines* BF , AD : I Book I.
say that the triangle ABC is equal to the triangle DEF .

For let AD be produced towards both parts to G and H ; and through B let BG be drawn parallel to CA ; and through the point F let FH be drawn parallel to DE .



Therefore (by the construction) each of the *figures* $GBCA$, $DEFH$ is a parallelogram: and (by prop. 36.) $GBCA$ is equal to $DEFH$, for they are upon equal bases BC , EF ; and between the same parallels BF , GH : and (by prop. 34.) the triangle ABC is half of the parallelogram $GBCA$; for the diameter AB cuts it in halves; also the triangle DEF is half of the parallelogram $DEFH$, for the diameter DF cuts it in halves: but (by com. not. 7.) the halves of equal magnitudes are equal: wherefore the triangle ABC is equal to the triangle DEF .

Therefore the triangles, which are upon equal bases and between the same parallels, are equal to one another. Which was to be demonstrated.

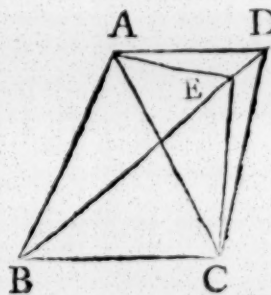
P R O P. XXXIX.

The equal triangles, which are upon the same base, and towards the same parts, are between the same parallels.

Let ABC , DBC be the equal triangles, which are upon the same base BC and towards the same parts: I say that they are between the same parallels: for let AD be joined: I say that AD is parallel to BC .

For if *it is* not, let AE be drawn (by prop. 31.) through the point A parallel to the straight line BC , and let EC be joined.

Wherefore (by prop. 37.) the triangle ABC is equal to the triangle EBC , for it is upon the same base with it the *straight line* BC ; and between the same parallels BC , AE : but the triangle ABC is equal (by sup.) to the triangle DBC ; therefore (by com. not. 1.) the triangle DBC is equal to the triangle EBC , the greater to the less (by com. not.



Book I. 9.) which is impossible: wherefore AE is not parallel to BC: certainly we shall demonstrate in like manner, that neither is any other *straight line* except AD; therefore AD is parallel to BC.

Wherefore the equal triangles, which are upon the same base, and towards the same parts, are between the same parallels. Which was to be demonstrated.

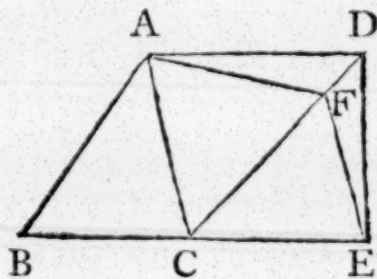
P R O P. XL.

The equal triangles, which are upon equal bases, and towards the same parts, are between the same parallels.

Let ABC, CDE be the equal triangles, which are upon equal bases the *straight lines* BC, CE; and towards the same parts: I say that they are between the same parallels; for let AD be joined I say that AD is parallel to BE.

For if *it is not*; let FA be drawn through the *point* A (by prop. 31.) parallel to BE, and let FE be joined.

Therefore (by prop. 38.) the triangle ABC is equal to the triangle FCE; for they are upon the equal bases BC, CE; and between the same parallels BE, AF; but (by the sup.) the triangle ABC is equal to the triangle DCE; and therefore (by com. not. 1.) the triangle DCE is equal to the triangle FCE; the greater to the less (by com. not. 9.) which is impossible: wherefore AF is not parallel to BE: certainly we shall demonstrate in the same manner that neither *is* any other except the *straight line* AD: therefore AD is parallel to BE.



Wherefore the equal triangles, which are upon equal bases, and towards the same parts, are between the same parallel lines. Which was to be demonstrated.

P R O P. XLI.

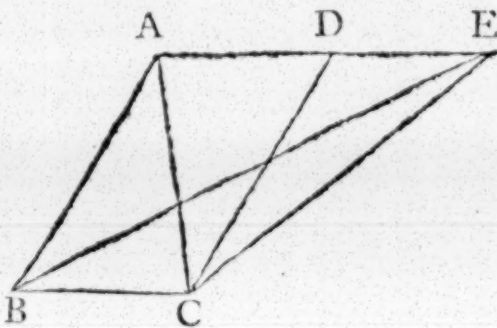
If a parallelogram have the same base with a triangle; and be between the same parallel lines; the parallelogram will be double of the triangle.

For

For let the parallelogram ABCD have the same base with the triangle EBC, the *straight line* BC; and let them be between the same parallels BC, AE: I say that the parallelogram ABCD is double of the triangle BEC. Book I.

For let AC be joined.

Certainly the triangle ABC is equal to the triangle EBC (by prop. 37.); for it is upon the same base with it, the *straight line* BC and between the same parallels BC, AE; but the parallelogram ABCD (by prop. 34.) is double of the triangle ABC; for the diameter AC cuts it in halves: wherefore the parallelogram ABCD is also double of the triangle EBC.



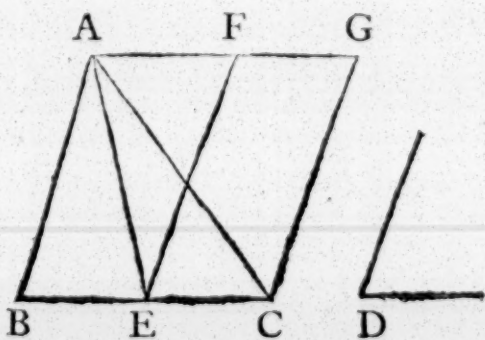
If, therefore, a parallelogram have the same base with a triangle; and be between the same parallels; the parallelogram will be double of the triangle. Which was to be demonstrated.

P R O P. XLII.

To make a parallelogram equal to a given triangle, in a given rectilineal angle.

Let ABC be the given triangle; and D the given rectilineal angle: it is required to make a parallelogram equal to the triangle ABC, in a rectilineal angle equal to D.

Let BC be cut in halves (by prop. 10.) at the *point* E; and let AE be joined; and let the angle CEF be made with the *straight line* EC and at the *point* E in it equal to the angle D (by prop. 23.); and through the *point* A let AG be drawn parallel to EC; and through the *point* C (by prop. 31.) let CG be drawn parallel to EF: therefore FECG is a parallelogram.



And because BE is equal to EC, the triangle ABE (by prop. 38.) is also equal to the triangle AEC; for they are upon equal bases BE,

Book I.

BE, EC and between the same parallel lines BC, AG; wherefore the *triangle* ABC is double of the triangle AEC: but the parallelogram FECG is also (by prop 41.) double of the triangle AEC: for they have the same base, and are between the same parallels; therefore (by com. not. 6.) the parallelogram FECG is equal to the triangle ABC; and has the angle CEF equal to the given angle D.

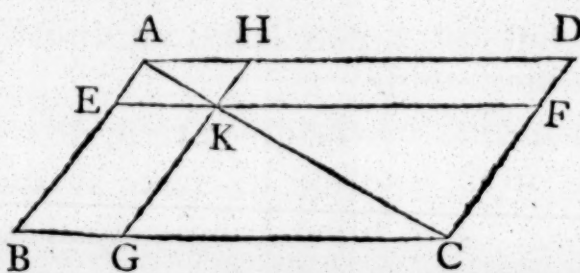
Therefore a parallelogram the *parallelogram* FECG has been made equal to the given triangle ABC, in an angle the angle CEF, which is equal to the angle D. Which was to be done.

P R O P. XLIII.

The complements of the parallelograms about the diameter of every parallelogram are equal to one another.

Let ABCD be a parallelogram, and AC the diameter of it; and let EH, FG be the parallelograms about AC; and the *parallelograms* BK, KD those which are called complements: I say that the complement BK is equal to the complement KD.

For since ABCD is a parallelogram, and AC the diameter of it; the triangle ABC is (by prop. 34.) equal to the triangle ADC: **again** because EKHA is a parallelogram, and AK the diameter of it; the triangle AEK is (by prop. 34.) equal to the triangle AHK: certainly for the same reason also the triangle KFC is equal to the triangle KGC: because therefore the triangle AEK is equal to the triangle AHK, and KFC to KGC; the triangle AEK together with the triangle KGC is equal (by com. not. 2.) to the triangle AHK together with the triangle KFC; but the whole triangle ABC is equal to the whole triangle ADC; therefore (by com. not. 3.) the *remainder*, the complement BK is equal to the remainder the complement KD.



Wherefore the complements of the parallelograms about the diameter of every parallelogram are equal to one another. Which was to be demonstrated.

P R O P.

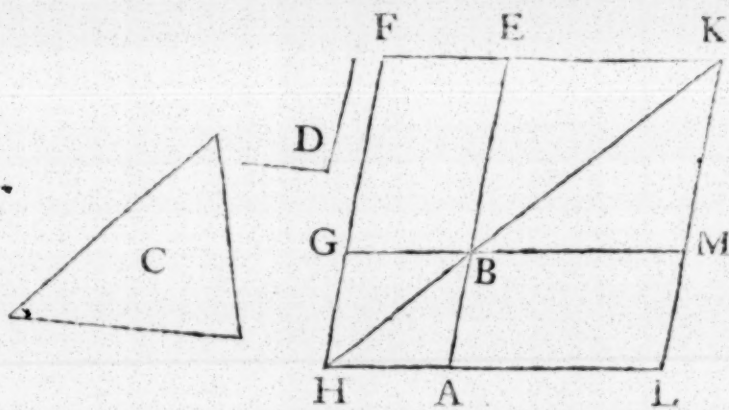
PROP. XLIV.

Book I.

To apply a parallelogram, to a given straight line, equal to a given triangle, in a given rectilineal angle.

Let the given straight line be AB ; but the given triangle the *triangle* C ; and the given rectilineal angle, the *angle* D ; it is required to apply a parallelogram to the given straight line AB , equal to the given triangle C ; in an angle equal to D .

Let the parallelogram $BEFG$ be made (by prop. 42.) equal to the triangle C , in an angle EBG which is equal to D : and let it be placed in such a manner that BE may be in a straight line with AB ; and let FG be produced to H ; and through



the *point* A , let AH be drawn (by prop. 31.) parallel to either of the *straight lines* BG or EF ; and let HB be joined.

And because the straight line HF hath fallen upon the parallels AH , EF ; therefore the angles (by prop. 29.) AHF , HFE are equal to two right angles: wherefore the angles BHG , GFE are less than two right angles: but (by com. not. 11.) those *straight lines*, which are produced indefinitely from angles less than two right angles, do meet *each other*; therefore HB , FE being produced will meet: let them be produced, and let them meet in the *point* K ; and through the point K , let KL be drawn parallel to either of the *lines* EA , FH ; and let GB , HA be produced to the points M , L .

Therefore $HLKF$ is a parallelogram, and HK its diameter; and AG , ME are the parallelograms about HK ; and LB , BF the *parallelograms* called complements: wherefore (by prop. 43.) LB is equal to BF : but BF is also equal to the triangle C ; and therefore (by com. not. 1.) LB is equal to C : and since the angle GBE is equal (by prop. 15.) to ABM , but GBE is equal (by const.) to D ; also (by com. not. 1.) ABM is equal to the angle D .

VOL. I.

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Book I.

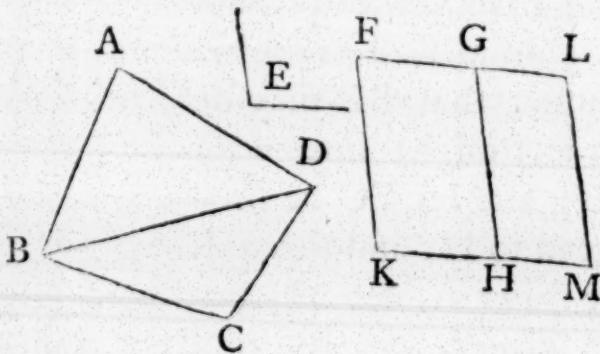
Wherefore a parallelogram LB hath been applied to the given straight line AB, equal to the given triangle C; in an angle ABM, which is equal to the *angle* D. Which was to be done.

PROP. XLV.

To make a parallelogram equal to a given rectilineal *figure* in a given rectilineal angle.

Let the given rectilineal *figure* be ABCD; and E the given rectilineal angle: it is required to make a parallelogram equal to the rectilineal *figure* ABCD in an angle equal to E.

For let DB be joined; and (by prop. 42.) let the parallelogram FH be made equal to the triangle ABD, in the angle HKF, which is equal to E; and (by prop. 44.) let the parallelogram GM, equal to the triangle DBC, be applied to the straight line GH, in the angle GHM, which is equal to E.



And because the angle E is equal to either of the angles HKF, GHM; therefore also (by com. not. 1.) HKF is equal to GHM: let the common angle KHG be added; therefore (by com. not. 2.) FKH, KHG are equal to KHG, GHM: but (by prop. 29.) FKH, KHG are equal to two right angles; wherefore also KHG, GHM are equal to two right angles: to a certain straight line GH, and to a point in it H, the two straight lines KH, HM not lying towards the same parts, make the adjacent angles equal to two right angles; therefore (by prop. 14.) KH is in a straight line with HM: and since the straight line GH hath fallen upon the parallels KM, FG; the alternate angles MHG, HGF are equal (by prop. 29.): let the common angle HGL be added: therefore the *angles* MHG, HGL are equal to HGF, HGL; but (by prop. 29.) MHG, HGL are equal to two right angles; therefore also HGF, HGL are equal to two right angles; wherefore (by prop. 14.) FG is in a straight line with GL. And because KF is equal and also

also parallel to GH; but GH is equal and also parallel to ML; therefore (by com. not. 1. and prop. 30.) KF is equal and parallel to ML; and the straight lines KM, FL join them: and (by prop. 33.) KM, FL are equal and parallel. Wherefore KFLM is a parallelogram; and because the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM; therefore the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM.

Wherefore a parallelogram KFLM is made equal to a given rectilineal figure ABCD in an angle, the angle FKM, which is equal to the given angle E. Which was to be done.

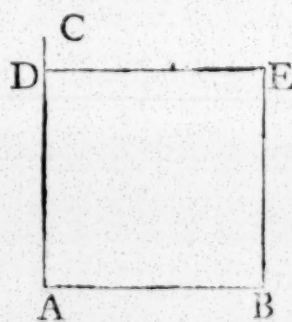
P R O P. XLVI.

To describe a square upon a given straight line.

Let the given straight line be AB: it is required to describe a square upon the straight line AB.

Let AC be drawn (by prop. 11,) at right angles to the straight line AB from the point A in it; and (by prop. 3.) make AD equal to AB: and let DE be drawn, (by prop. 31.) through the point D, parallel to AB; and let BE be drawn through the point B parallel to AD.

Therefore the figure ADEB is a parallelogram; wherefore (by prop. 34.) AB is equal to DE; and AD to BE; but also AB is equal to AD (by const.) therefore the four straight lines BA, AD, DE, EB are equal to one another: wherefore the parallelogram ADEB is equilateral; I say that it is also rectangular: For because the straight line AD has fallen upon the two parallels AB, DE; therefore (by prop. 29.) the angles BAD, ADE are equal to two right angles; but BAD is a right angle (by const.); therefore ADE is a right angle; but the opposite sides and also the angles of parallelogram spaces are equal to one another (by prop. 34.); therefore each of the opposite angles is a right angle, the angles ABE, BED; wherefore the figure ADEB is rectangular; but it has been demonstrated to be equilateral also.



Book I.

Therefore (by def. 30.) it is a square; and it hath been described upon the straight line AB. Which was to be done.

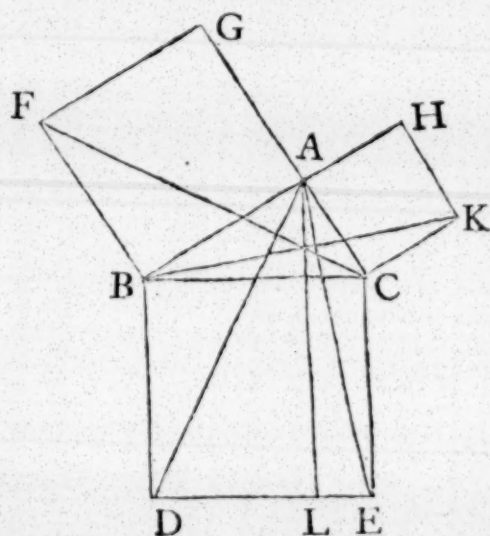
PROP. XLVII.

In right angled triangles, the square of the side subtending the right angle, is equal to the squares of the sides containing the right angle.

Let ABC be a right angled triangle, having the angle BAC a right angle: I say that the square of BC is equal to the squares of BA, AC.

For (by prop. 46.) let the square BDEC be described upon BC; and the squares GB, HC upon BA, AC, and through the point A let AL be drawn parallel either to BD or CE; and let AD, FC be joined.

And because each of the angles BAC, BAG is (by const. and sup.) a right angle; to a certain line BA, and to the point A in it, two straight lines AC, AG, not lying towards the same parts, make the adjacent angles equal to two right angles; therefore (by prop. 14.) AC is in a straight line with AG: certainly for the same reason also AB is in a straight line with AH: and since the angle DBC is equal to the angle FBA (by com. not. 10.), for each



of them is a right angle, let the common angle ABC be added; therefore (by com. not. 2.) the whole angle DBA is equal to the whole angle FBC; and because the two straight lines DB, BA are equal to the two CB, BF, each to each (being sides of the same square) and the angle DBA is equal to the angle FBC; therefore (by prop. 4.) the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC; and the parallelogram BL is double of the triangle ABD (by prop. 41.); for they have the same base BD and are between the same parallels BD, AL: but the square GB is double of the triangle FBC; for again, they have the

the same base FB, and are between the same parallels FB, GC; Book I.
 but the doubles of equal magnitudes are equal to one another (by {
 com. not 6.) wherefore also the parallelogram BL is equal to the
 square GB: Certainly in the same manner it will be demonstrated,
 having joined AE, BK, that the parallelogram CL is equal to the
 square HC; therefore the whole square DBCE is equal to the two
 squares GB, HC; and BDEC is the square described upon BC;
 and GB, HC the *squares described* upon BA, AC: wherefore the
 square BE *described* upon the side BC is equal to the squares upon
 the sides BA, AC.

Wherefore in right angled triangles, the square of the side sub-
 tending the right angle is equal to the squares of the sides, con-
 taining the right angle. Which was to be demonstrated.

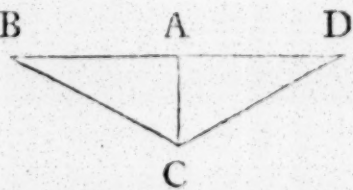
P R O P. XLVIII.

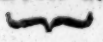
If the square of one of the sides of a triangle be equal to the
 squares of the two remaining sides of the triangle; the angle con-
 tained by the two remaining sides of the triangle is a right angle.

For let the square of BC one side of the triangle ABC be equal
 to the squares of the sides BA, AC: I say that the angle BAC is a
 right angle.

For from the point A let the straight line AD be drawn at right
 angles to AC; and make AD (by prop. 3.) equal to BA; and
 join DC.

And because DA is equal to AB, the
 square of AD is equal to the square of AB;
 let the common square of AC be added;
 therefore (by com. not. 2.) the squares of
 DA and AC are equal to the squares of BA and AC: but the
 squares of DA and AC are (by prop. 47.) equal to the square of
 DC; for DAC is (by const.) a right angle; and the square of BC
 is equal to the squares of BA and AC, for this is supposed: where-
 fore (by com. not. 1.) the square of DC is equal to the square of
 BC; so that also the side DC is equal to BC; and since AD is
 equal to AB and AC common; certainly the two AD, AC are equal
 to



Book I.  to the two AB, AC, and the base DC is equal to the base BC; therefore the angle DAC is equal to the angle BAC; but DAC is a right angle; therefore the *angle* BAC is a right angle.

Wherefore if the square of one of the sides of a triangle be equal to the squares of the two remaining sides of the triangle; the angle contained by the two remaining sides of the triangle is a right angle. Which was to be demonstrated.

THE
ELEMENTS
OF
EUCLID.
BOOK II.

DEFINITIONS.

1. **E**VERY right angled parallelogram is said to be contained by the two straight lines containing the right angle. Book II.
2. Of every parallelogram space, let any one of the parallelograms about the diameter of it, together with the two complements, be called a GNOMON.

PROP. I.

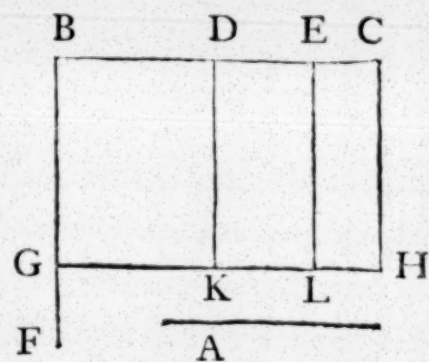
If there be two straight lines, and one of them be cut into any number of segments; the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and each of the segments *of the divided line*.

Let there be two straight lines A, BC, and let BC be cut as it may happen in the points D and E: I say that the rectangle contained

Book II. { tained by A and BC is equal to the rectangle contained by A and BD, and to that *contained* by A and DE, and besides to that *contained* by A and EC.

For let BF be drawn from the *point* B at right angles to BC ; and let BG be made (by prop. 3. B. 1.) equal to A ; and let GH be drawn, through the point G parallel to BC (by prop. 31. B. 1.) ; and let DK, EL, CH be drawn through the points D, E, C parallel to BG.

Certainly the *rectangle* BH is equal to the rectangles BK, DL, EH ; and BH is the rectangle *contained* by A and BC ; for (by def. 1. B. 2.) it is contained by GB, BC ; but BG is (by const.) equal to A : and BK is a *rectangle contained* by A and BD ; for (by def. 1.) it is contained by GB, BD ; and GB is equal to A : but the rectangle DL is *contained* by A and DE ; for DK, that is BG (by prop. 34. B. 1.) is equal to A : and also in like manner EH is a *rectangle* contained by A and EC , therefore the rectangle contained by A and BC is equal to the *rectangles contained* by A and BD ; and A and DE and also A and EC.



Wherefore if there be two straight lines, and one of them be cut into any number of segments ; the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and each of the segments *of the divided line*. Which was to be demonstrated.

PROP. II.

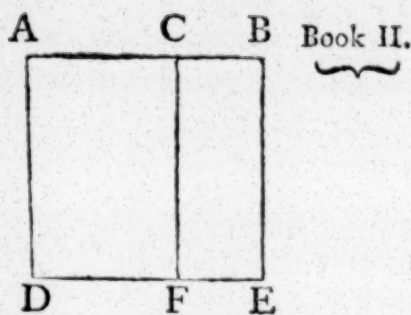
If a straight line be cut as it may happen, the rectangles contained by the whole and each of the segments are equal to the square of the whole *line*.

For let the straight line AB be cut as it may happen in the point C : I say that the rectangle *contained* by AB and BC ; together with the rectangle contained by BA and AC is equal to the square of AB.

For let the square ADEB be described upon AB (by 46. 1.) ; and let CF be drawn (by 31. 1.) parallel either to AD or BE.

Certainly

Certainly the *rectangle* AE is equal to the *rectangles* AF and CE : and the *rectangle* AE is the square of AB ; and the *rectangle* AF is a rectangle contained by BA, AC ; for (by def. 1. 2.) it is contained by DA, AC ; and AD is equal to AB (being sides of a square) : and the rectangle CE is contained by AB, BC ; for EB is equal to AB : wherefore the rectangle contained by BA, AC together with that contained by AB, BC is equal to the square of AB.



Book II.

If therefore a straight line be cut as it may happen, the rectangles contained by the whole and each of the segments are equal to the square of the whole *line*.

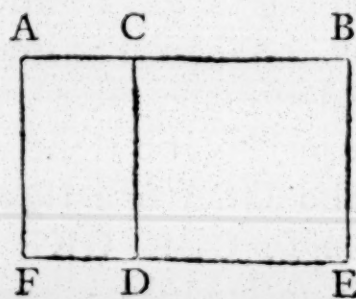
PROP. III.

If a straight line be cut as it may happen, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the *two* segments ; and to the square of the forementioned segment.

For let the straight line AB be cut as it may happen in the point C : I say that the rectangle contained by AB, BC is equal to the rectangle contained by AC, CB ; together with the square of BC.

For let CDEB the square of BC be described (by 46. 1.) ; and let ED be produced to F : and through the *point* A let AF be drawn (by 31. 1.) parallel to either of the *straight lines* CD, BE.

Certainly the rectangle AE is equal to the *rectangles* AD, CE ; and the *rectangle* AE is a rectangle contained by AB, BC ; for (by def. 1. B. 2.) it is contained by AB, BE ; but BE is equal to BC (being sides of a square) : And the rectangle AD is contained by AC, CB ; for DC is equal to CB ; but the *rectangle* DB is the square of CB : there-



fore the rectangle contained by AB, CB is equal to the rectangle contained by AC, CB ; together with the square of BC.

If therefore a straight line be cut as it may happen, the rectangle, contained by the whole and one of the segments, is equal to the

Book II. rectangle contained by the *two* segments; and to the square of the forementioned segment. Which was to be demonstrated.

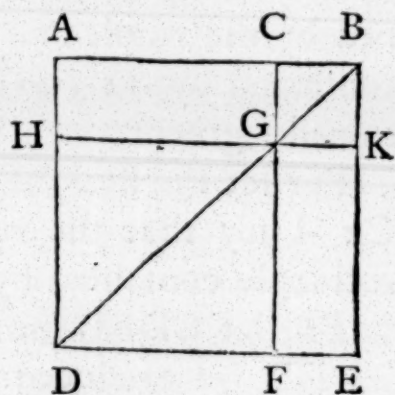
P R O P. IV.

If a straight line be cut as it may happen, the square of the whole *line* is equal to the squares of the *two* segments, and to the rectangle, contained by the *two* segments, *taken twice*.

For let the straight line AB be cut as it may happen in the *point* C: I say that the square of AB is equal to the squares of AC, CB and the rectangle contained by AC, CB *taken twice*.

For let the square ADEB be described upon AB (by 46. 1.); and let BD be joined, and through the *point* C let CGF be drawn parallel to either of the *lines* AD, BE; and through the point G let HK be drawn parallel to either of the *lines* AB, DE.

And because CF is parallel to AD and BD hath fallen upon them; the outward angle BGC is equal to the inward and opposite ADB: but the angle ADB (by 5. 1.) is equal to ABD, because the side BA is equal to AD; therefore (by com. not. 1.) the angle CGB is equal to GBC; so that also (by 6. 1.) the side BC is equal to the side CG: but (by 34. 1.) CG is equal to BK and CB to GK; wherefore also GK is equal to KB: wherefore CGKB is equilateral; I say also that it is rectangular: For because CG is parallel to BK and CB hath fallen upon them; therefore (by 29. 1.) the angles KBC, BCG are equal to two right angles; but KBC is a right angle; wherefore also GCB is a right angle; so that also (by 34. 1.) the opposite angles CGK, GKB are right angles: wherefore the *figure* CGKB is rectangular; and it has been demonstrated to be equilateral; therefore (by def. 30. 1.) it is a square and it is *described* upon BC: Certainly for the same reason also the *figure* HF is a square; and it is *described* upon HG that is upon AC: therefore the *figures* HF, CK are the squares of AC, CB. And because AG is equal to GE (by 43. 1.); and AG is the *rectangle* contained by AC, CB; for GC is



GC is equal to CB; and GE is therefore equal to the *rectangle con-* Book II.
tained by AC, CB: wherefore AG, GE are equal to the *rectangle*
contained by AC, CB *taken twice*; but also HF, CK are the squares
 of AC, CB; therefore the four *figures* HF, CK, AG, GE are
 equal to the squares of AC, CB and the *rectangle contained* by AC,
 CB *taken twice*: but HF, CK, AG, GE are the whole *figure*
 ADEB, which is the square of AB; wherefore the square of AB
 is equal to the squares of AC, CB and the *rectangle contained* by
 AC, CB *taken twice*.

Wherefore if a straight line be cut as it may happen, the square
 of the whole *line* is equal to the squares of the *two* segments, and
 to the *rectangle* contained by the *two* segments *taken twice*.

Another Demonstration.

I say that the square of AB is equal to the squares of AC, CB
 and to the rectangle contained by AC, CB *taken twice*.

For, making use of the same figure, because BA is equal to AD,
 (by 5. 1.) the angle ABD is equal to the *angle* ADB; and since
 (by 32. 1.) the three angles of every triangle are equal to two right
 angles; therefore the three angles of the triangle ABD viz. ABD,
 ADB, BAD are equal to two right angles; but BAD is a right
 angle (being an angle of a square); therefore the remaining angles
 ABD, ADB are equal to one right angle; and they are equal:
 each, therefore, of the angles ABD, ADB is the half of a right
 angle: but BCG is a right angle (by prop. 29. 1.) for it is equal
 to the inward and opposite *angle* at A; therefore the remainder
 CGB is the half of a right angle: wherefore the angle CGB is
 equal to CBG; so that also (by 6. 1.) the side BC is equal to the
side CG; but (by 34. 1.) CB is equal to KG, and CG to BK:
 wherefore the *figure* CK is equilateral; but it has a right angle, the
angle CBK: wherefore CK is a square; and it is *described* upon
 CB. Certainly for the same reason also HF is a square; and is equal
 to the square of AC: wherefore CK, HF are squares, and are
 equal to the squares of AC, CB: and because AG is equal to GE
 (by 43. 1.) and AG is the *rectangle contained* by AC, CB; for CG
 is equal to CB; and EG is equal to the *rectangle contained* by AC,
 CB; therefore the *figures* AG, GE are equal to the *rectangle con-*
tained

Book II. *tained* by AC, CB; *taken* twice: but CK, HF are equal to the *squares* of AC, CB; therefore the *figures* CK, HF, AG, GE are equal to the *squares* of AC, CB and the *rectangle* contained by AC, CB *taken* twice: But the *squares* CK, HF and the rectangles AG, GE are the whole *square* AE, which is the square of AB.

Wherefore the square of AB is equal to the squares of AC, CB; and to the rectangle contained by AC, CB *taken* twice. Which was to be demonstrated.

Cor. Certainly from these *demonstrations* it is manifest that in square spaces the parallelograms about the diameter are squares.

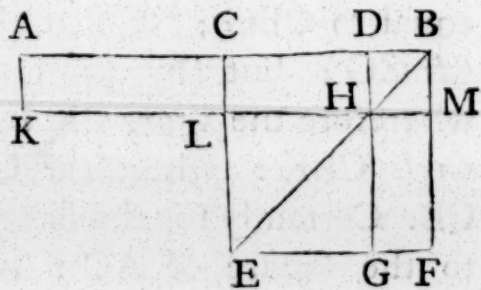
P R O P. V.

If a straight line be cut into equal and unequal *segments*; the rectangle contained by the unequal segments of the whole *line*, together with the square of the *line* between the *points* of section is equal to the square of half *the line*.

For let any straight line AB be cut into equal *segments* at the *point* C; and into unequal *segments* at the *point* D; I say that the rectangle contained by AD, DB together with the square of CD is equal to the square of CB.

For let the square CEFB be described upon BC; and let BE be joined; and let DHG be drawn, through the *point* D, parallel to either of the *lines* CE, BF; and again through the *point* H let KLM be drawn parallel to either of the *lines* CB, EF; and again through the *point* A let AK be drawn parallel to either of the *lines* CL, BM.

And since the complement CH is equal to the complement HF; let DM *which is* common be added; therefore the whole CM is equal to the whole DF: but CM is equal to AL (by prop. 36. 1.) because AC is equal to CB: therefore also (by com. not. 1.) AL is equal to DF; let CH *which is* common be added; therefore the whole AH is equal to DF and DL: but AH is the *rectangle* contained by AD, DB for DH is equal to DB (by cor. to 4. 2.): but FD,



FD, DL or FDL is called a Gnomon (by def. 2. 2.); therefore the Gnomon FDL is equal to the *rectangle* contained by AD, DB: Book II. Let LG which is common be added; which (by Cor. to 4. 2.) is equal to the square of CD: wherefore the Gnomon FDL and the *square* LG is equal to the rectangle contained by AD, DB and the square of CD: but the Gnomon FDL and the square LG is the whole square CEFB, which is the *square* of CB: Wherefore the rectangle contained by AD, DB together with the square of CD is equal to the square of CB.

Wherefore if a straight line be cut into equal and unequal *segments*; the rectangle contained by the unequal segments of the whole *line*, together with the square of the *line* between the *points* of section, is equal to the square of half *the line*. Which was to be demonstrated.

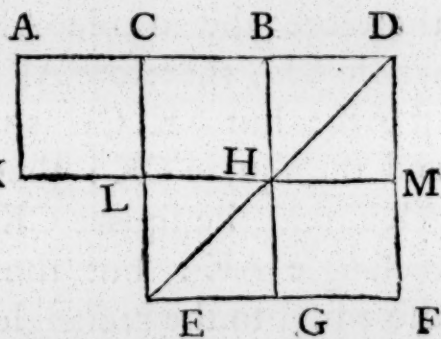
P R O P. VI.

If a straight line be cut in halves, and any straight line be added to it in a straight line; the rectangle contained by the whole with the part produced, and the part produced, together with the square of half the *line*; is equal to the square described upon the compounded *line*, that is of half the line and the part produced, as upon one *line*.

For let any straight line AB be cut in halves at the point C, and let any straight line BD be added to it in a straight line: I say that the rectangle contained by AD, DB together with the square of BC is equal to the square of CD.

For let the square CEFD be described (by 46. 1.) upon CD; and let DE be joined; and through the point B let (by 31. 1.) BHG be drawn parallel to either of the *lines* CE, DF; and through the point H, let KLM be drawn parallel to either of the *lines* AD, EF; and farther through the *point* A let AK be drawn parallel to either of the *lines* CL, DM.

Therefore because AC is equal to CB, the *rectangle* AL (by prop. 36. 1.) is equal to the *rectangle* CH; but CH is also equal (by 43. 1.) to HF; wherefore AL is equal to HF; let CM *which* is common be added, therefore the whole AM is equal to CM, MG that is (by def. 2. 2.)



Book II. to the Gnomon CMG : but AM is the *rectangle* contained by AD, DB (by def. 1. 2.) for DM is equal to DB (by cor. to 4. 2.) ; therefore the Gnomon CMG is equal to the rectangle contained by AD, DB : let GL *which is* common be added ; which is equal (by cor. to 4. 2.) to the square of CB : therefore the rectangle contained by AD, DB together with the square of CB is equal to the Gnomon CMG and the *square* LG : but the Gnomon CMG and the *square* LG is the whole square CEFD, which is the square of CD : wherefore the rectangle contained by AD, DB together with the square of CB is equal to the square of CD.

Wherefore if a straight line be cut in halves, and any straight line be added to it in a straight line ; the rectangle contained by the whole with the part produced and the part produced ; together with the square of half of the *line* ; is equal to the square described upon the compounded *line* ; that is of half *the line* and the part produced, as upon one *line*. Which was to be demonstrated.

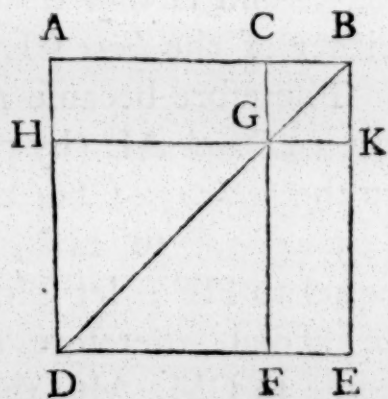
P R O P. VII.

If a straight line be cut as it may happen ; the *square* of the whole *line* and the *square* of one of the segments, these squares, taken together, are equal to the rectangle contained by the whole, and the said segments, *taken twice* and the square of the remaining segment.

For let any straight line AB be cut as it may happen in the point C ; I say that the squares of AB, BC are equal to the rectangle contained by AB, BC taken twice and the square of AC.

For let a square ADEB be described upon AB : and let the figure be constructed.

And because AG is equal to GE (by 43. 1.) ; let CK which is common be added : therefore the whole AK is equal to the whole CE ; therefore AK, CE are the double of AK : but AK, CE is the Gnomon AKF : and the square CK : therefore the Gnomon AKF and the square CK are double of the rectangle AK : but the double of AK is also equal to the rectangle contained by AB,



BC *taken twice*; for BK is (by cor. to 4. 2.) equal to BC: Wherefore the Gnomon AKF and the square CK are equal to the rectangle contained by AB, BC *taken twice*: Let HF which is common be added, which is the square of AC: therefore the Gnomon AKF and the squares CK and HF are the whole *square* ADEB and the *square* CK, which are the squares of AB, BC: wherefore the squares of AB, BC are equal to the rectangle contained by AB, BC *taken twice* together with the square of AC.

Wherefore if a straight line be cut as it may happen; the *square* of the whole *line* and the square of one of the segments, these squares, taken together, are equal to the rectangle contained by the whole and the said segment, taken twice; and the square of the remaining segment. Which was to be demonstrated.

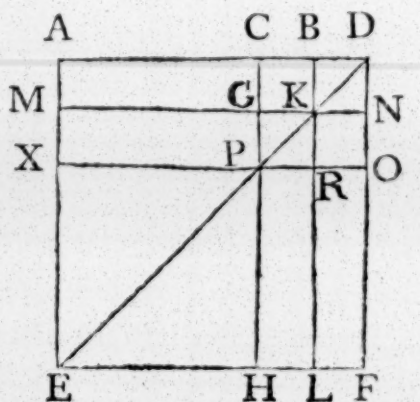
P R O P. VIII.

If a straight line be cut as it may happen; the rectangle contained by the whole *line* and one of the segments, *taken four times*, together with the square of the remaining segment, is equal to the square described upon the whole and the said segment, as upon one *line*.

For let any straight line AB be cut as it may happen in the point C: I say that the rectangle contained by AB, BC; *taken four times*, together with the square of AC is equal to the square described upon AB, BC; as upon one line.

For let BD be produced in a straight line with AB; and make BD equal to CB; and let the square AEFD be described upon AD: and let the double figure be constructed.

Therefore because CB is equal to BD; but (by 34. 1.) CB is equal to GK, and BD to KN; also (by com. not. 1.) GK is equal to KN. Certainly for the same reason PR is equal to RO: and because CB is equal to BD, and GK to KN; therefore also the *figure* CK is equal to BN (by 36. 1.), and GR to RN; but the *figure* CK is equal to RN (by 43. 1.); for they



are

Book II.

are the complements of the parallelogram CO : wherefore BN is also equal to GR (by con. not. 1.) : therefore the four *figures* the *rectangles* BN, KC, GR, RN are equal to one another ; the four therefore are quadruple of CK : Again because CB is equal to BD ; but BD (by cor. to 4. 2.) is equal to BK, that is to CG (by 34 1.) ; and CB to GK, that is, is equal to GP ; therefore CG is equal to GP : And because CG is equal GP ; and PR to RO ; also the *rectangle* AG is equal (by 36. 1.) to MP : and PL to RF ; but MP and PL are equal (by 43. 1.), for they *are* the complements of the parallelogram ML : therefore also AG is equal to RF : wherefore the four *figures*, the *rectangles* AG, MP, PL, RF are equal to one another : therefore the four *figures* are quadruple of AG : but it has been demonstrated that the four CK, BN, GR, RN are quadruple of CK ; therefore the eight which contain the Gnomon AOH are quadruple of AK ; and because AK is the *rectangle contained* by AB, BD ; for BK is equal to BD ; therefore the *rectangle* contained by AB, BD *taken* four times is equal to the Gnomon AOH : let XH which is common be added, which is equal to the square of AC : wherefore the *rectangle* contained by AB, BD *taken* four times, together with the square of AC is equal to the Gnomon AOH and the *square* XH : but the Gnomon AOH and the *square* XH is the whole square AEFD, which is the *square* of AD : therefore the *rectangle* contained by AB, BC *taken* four times together with the square of AC is equal to the square of AD, that is to the square described upon AB and BC as upon one *line*.

Wherefore if a straight line be cut as it may happen ; the *rectangle* contained by the whole *line*, and one of the segments ; taken four times ; together with the square of the remaining segment, is equal to the square described upon the whole and the said segment, as upon one *line*. Which was to be demonstrated.

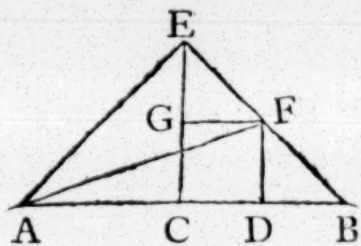
P R O P. IX.

If a straight line be cut into equal and unequal *segments* ; the squares, of the unequal segments of the whole *line*, is double both of the square of half *the line* and of the *line* between the *points* of section.

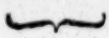
For

For let the straight line AB be cut into equal segments at the *point* C, and into unequal ones at the *point* D: I say that the squares of AD, DB are double of the squares of AC, CD. Book II.

For from the *point* C, let CE be drawn at right angles to AB; and let it be made equal to either of the lines AC, CB; and let EA, EB be joined; and through the point D let DF be drawn parallel to EC; and through the point F, let FG be drawn parallel to AB, and let AF be joined.



And because AC is equal to CE, the angle EAC (by 5. 1.) is equal to AEC: and since the angle at C is a right angle, therefore the remaining angles AEC, EAC are equal to one right angle (by 32. 1.): therefore each of the angles AEC, EAC is the half of a right angle. Certainly for the same reason also each of the angles CEB, EBC is the half of a right angle; wherefore the whole angle AEB is a right angle: and because GEF is half a right angle, and EGF a right angle (by. 29. 1.) for it is equal to the inward and opposite ECB; therefore the remainder EFG is half a right angle (by 32. 1.); therefore the angle GEF is equal to the angle EFG; so that also (by 6. 1.) the side EG is equal to the side GF: Again because the angle at B is half a right angle, and FDB a right angle (by 29. 1.) for again it is equal to the inward and opposite ECB; therefore the remaining angle BFD is half a right angle: therefore the angle at B is equal to the angle DFB; so that also the side DF is equal (by 6. 1.) to the side DB: And because AC is equal to CE: the square of AC is also equal to the square of CE; therefore the squares of AC, CE; are double of the square of AC: but the square of EA is equal to the squares of AC, CE (by 47. 1.) for the angle ACE is a right angle: wherefore the square of EA is double of the square of AC: Again because EG is equal to GF, the square of EG is equal to the square of GF; therefore the squares of EG, GF are double of the square of GF; but the square of EF is equal to the squares of EG, GF (by 47. 1.); therefore the square of EF is double of the square of GF; but (by 34. 1.) GF is equal to CD; therefore the square of EF is double the square of CD: but the square of AE is also the

Book II.  double of the *square* of AC; wherefore the squares of AE, EF are double of the squares of AC, CD; but the square of AF is equal (by 47. 1.) to the squares of AE, EF; for the angle AEF is a right angle; therefore the square of AF is double of the *squares* of AC, CD: but the *squares* of AD, DF are equal (by 47. 1.) to the *square* of AF, for the angle at D is a right angle; therefore the *squares* of AD, DF are double of the squares of AC, CD: but DF is equal to DB; wherefore the squares of AD, DB are double of the squares of AC, CD.

Wherefore if a straight line be cut into equal and unequal *segments*; the squares of the unequal segments of the whole *line*, is double both of the square of half *the line*, and of the *line* between the *points* of section. Which was to be demonstrated.

P R O P. X.

If a straight line be cut in halves, and any straight line be added to it in a straight line; the square of the whole with the part produced, and the square of the part produced, these *squares* taken both together are double both of the square of half the line, and of the square described upon the compounded *line*, that is of half *the line* and the part produced, as upon one *line*.

For let any straight line AB be cut in halves at the *point* C; and let any straight line BD be added to it in a straight line; I say that the squares of AD, DB are double of the squares of AC, CD.

For from the point C let CE be drawn at right angles to AB (by 11. 1.) and let it be made equal to either of the lines AC, CB; and let AE, EB be joined; and through the *point* E let EF be drawn (by 31. 1.) parallel to AD; and through the *point* D let DF be drawn parallel to CE: And because a certain straight line EF hath fallen upon the parallel straight lines EC, FD; (by 29. 1.) the angles CEF, EFD are therefore equal to two right angles; therefore the angles FEB, EFD are less than two right angles; but (by com. not. 11.) straight lines produced from *angles* less than two right angles meet *one another*; therefore EB, FD being produced towards the parts B, D will meet *one another*; let them be produced and meet at the *point* G; and let AG be joined.

And

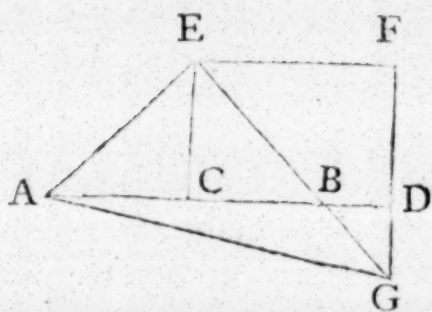
And because AC is equal to CE, (by 5. 1.) the angle AEC is equal to the *angle* EAC; and the *angle* at C is a right angle; wherefore each of the angles EAC, AEC is half a right angle.

Certainly for the same reason also, each of the angles CEB, EBC is half a right angle ; wherefore the angle AEB is a

right angle ; and since EBC is half a right angle ; therefore also (by 15. 1.) DBG is half a right angle ; but (by 29. 1.) BDG is a right angle, for it is equal to DCE, for *they are* alternate ; wherefore the remaining angle DGB is (by 32. 1.) half a right angle ; therefore the angle DGB is equal to the angle DBG ; so that also (by 6. 1.) the side BD is equal to the side GD : Again because EGF is half a right angle ; and the angle at F (by 34. 1.) is a right angle, for it is equal to the opposite angle at C : wherefore the remaining angle FEG is half a right angle ; wherefore the angle EGF is equal to the angle FEG ; so that (by 6. 1.) the side GF is equal to the side EF. And because EC is equal to CA ; also the square of EC is equal to the square of CA ; therefore the squares of EC, CA are double of the square of CA ; but (by 47. 1.) the *square* of EA is equal to the *squares* of EC, CA ; therefore the square of EA is double of the square of AC. Again, because GF is equal to EF, also the square of FG is equal to the square of EF ; therefore the *squares* of FG, FE are double of the *square* of FE ; but the square of EG is equal to the squares of EF, FG ; wherefore the *square* of EG is double of the square of EF ; but EF is (by 34. 1.) equal to CD ; therefore the square of EG is double of the *square* of CD ; but the *square* of EA has been also demonstrated to be double of the *square* of AC ; wherefore the squares of AE, EG are double of the squares of AC, CD : but the square of AG is (by 47. 1.) equal to the squares of AE, EG ; therefore the *square* of AG is double of the *squares* of AC, CD : but the *square* of AG is equal to the *squares* of AD, DG ; wherefore the squares of AD, DG ; are double of the squares of AC, CD : but DG is equal to DB ; therefore the squares of AD, DB are double of the squares of AC, CD.

H 2

Where-



Book II.

Wherefore if a straight line be cut in halves, and any straight line be added to it in a straight line; the square of the whole with the part produced, and the square of the part produced, these *squares* taken both together are double both of the square of half the line, and of the square described upon the compounded *line*, that is of half *the line* and the part produced, as upon one *line*. Which was to be demonstrated.

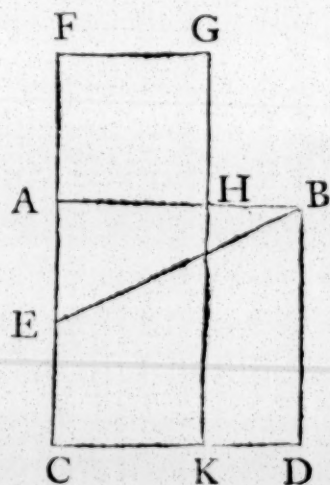
PROP. XI.

To cut a given straight line, so that the rectangle contained by the whole and one of the segments may be equal to the square of the remaining segment.

Let AB be the given straight line; it is required to cut the *line* AB, so that the rectangle contained by the whole and one of the segments, may be equal to the square of the remaining segment.

For let the square ABDC be described upon AB; and let AC be cut in halves in the point E; and let BE be joined; and let CA be produced to F; and let EF be made equal to BE; and let the square HF be described upon AF; and let GH be produced to K; I say that AB is cut in the point H so that the rectangle contained by AB, BH is equal to the square of AH.

For because the straight line AC is cut in halves in the *point* E, and AF is added to it; therefore the rectangle contained by CF, FA together with the square of AE is equal to the square of EF; but EF is equal to EB; wherefore the rectangle contained by CF, FA together with the square of AE is equal to the square of EB; but (by 47.1.) the *squares* of BA, AE are equal to the *square* of EB; for the angle at A is a right angle; therefore the *rectangle* contained by CF, FA together with the square of AE is equal to the *squares* of BA, AE; let the common square of AE be taken away; therefore the remainder, the *rectangle* contained by CF, FA is equal to the square of AB; and the *rectangle* FK is the *rectangle* contained by CF, FA; for FA is



is equal to FG ; and AD is the square of AB ; therefore FK is equal to AD ; let AK which is common be taken away; therefore the remainder FH is equal to HD ; and HD is the rectangle contained by AB, BH ; for AB is equal to BD : and FH is the square of AH ; wherefore the rectangle contained by AB, BH is equal to the square of AH . Book II.

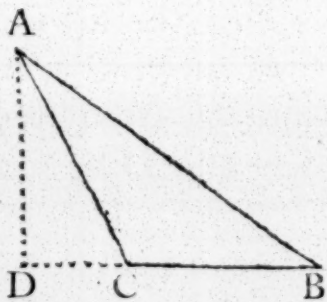
Wherefore the given straight line AB hath been cut in the point H , so that the rectangle contained by AB, BH is equal to the square of HA . Which was to be done.

P R O P. XII.

In obtuse angled triangles, the square, of the side subtending the obtuse angle, is greater, than the squares of the sides containing the obtuse angle by the rectangle twice taken contained by one of the sides about the obtuse angle upon which produced the perpendicular falls, and the line intercepted without, from the perpendicular to the obtuse angle.

Let ABC be an obtuse angled triangle, having an obtuse angle, the angle ACB ; and let the perpendicular AD be drawn from the point A , upon BC produced: I say that the square of AB is greater than the squares of AC, CB by the rectangle contained by BC, CD taken twice.

For since the straight line BD hath been cut as it may happen in the point C ; wherefore (by 4. 2.) the square of BD is equal to the squares of BC, CD and the rectangle contained by BC, CD taken twice: let the common square of AD be added; therefore the squares of BD, DA are equal to the squares of BC, CD, DA and to the rectangle contained by BC, CD taken twice: but the square of AB is (by 47. 1.) equal to the squares of BD, DA ; for the angle at D is a right angle; and the square of AC is equal to the squares of CD, DA ; wherefore the square of AB is equal to the squares of BC, CA and to the rectangle contained by BC, CD taken twice; so that the square of AB is greater than



Book II. than the squares of BC, CA by the rectangle contained by BC, CD *taken twice*.

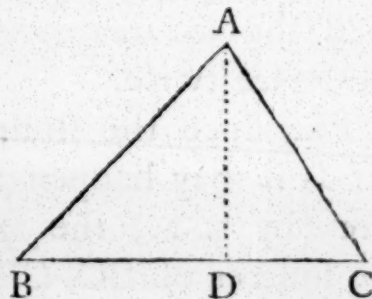
Wherefore in obtuse angled triangles the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle by the *rectangle twice taken* contained by one of the sides about the obtuse angle upon which produced the perpendicular falls, and the line intercepted without from the perpendicular to the obtuse angle. Which was to be demonstrated.

P R O P. XIII.

In acute angled triangles, the square of the side subtending the acute angle is less than the squares of the sides containing the acute angle, by the rectangle *taken twice*, contained by one of the sides about the acute angle upon which the perpendicular falls and the line intercepted within, from the perpendicular to the acute angle.

Let there be an acute angled triangle, the triangle ABC, having the angle at B an acute angle; and let the perpendicular AD be drawn from the point A upon BC: I say that the square of AC is less than the squares of CB, BA by the rectangle CB, BD *taken twice*.

For because the straight line CB hath been cut as it may happen in the point D; therefore the squares of CB, BD are equal to the rectangle CB, BD *taken twice* and to the square of DC (by 7. 2.): let the common square of AD be added: therefore the squares of CB, BD, DA are equal to the rectangle contained by CB, BD *taken twice*, and to the squares of CD, DA; but the square of AB is equal to the squares of BD, DA (by 47. 1.) for the angle at D is a right angle; and the square of AC is equal to the squares of CD, DA: wherefore the squares of CB, BA are equal to the rectangle contained by CB, BD, *taken twice* and the square of AC; so that the square of AC alone is less than the squares of CB, BA by the rectangle contained by CB, BD *taken twice*.



Where-

Wherefore in acute angled triangles, the square of the side subtending the acute angle is less than the squares of the sides containing the acute angle, by the rectangle *taken* twice, contained by one of the *sides* about the acute angle upon which the perpendicular falls, and the line intercepted within, from the perpendicular to the acute angle. Which was to be demonstrated. Book II.

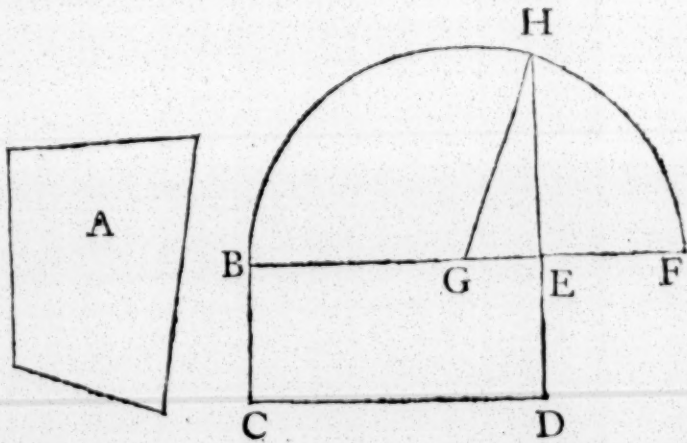
P R O P. XIV.

To make a square equal to a given rectilinear figure.

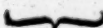
Let *A* be the given rectilinear figure; it is required to make a square equal to the given rectilinear figure *A*.

Let the right angled parallelogram *BD* be made (by 45. 1.) equal to the rectilinear figure *A*; if therefore *BE* be equal to *ED* what was required hath been done; for the square *BD* is made equal to the rectilinear figure *A*: but if not, one of them *BE*, *ED* is greater: let *BE* be the greater, and let it be produced to *F*, and make *EF* equal to *ED*; and let *FB* be cut in halves (by 10. 1.) in the point *G*; and with the center *G* and at the distance of one of the lines *GB*, *GF* let the semicircle *BHF* be described, and let *DE* be produced to *H*, and let *GH* be joined.

Therefore because the straight line *EF* hath been cut into equal segments at the point *G*, and into unequal segments at *E*; therefore (by 5. 2.) the rectangle contained by *BE*, *EF* together with the square of *EG* is equal to the square of *GF*; but *GF* is equal



to *GH*; therefore the rectangle contained by *BE*, *EF* together with the square of *EG* is equal to the square of *GH*; but the squares of *HE*, *EG* (by 47. 1.) are equal to the square of *GH*; wherefore the rectangle contained by *BE*, *EF* together with the square of *GE* is equal to the squares of *HE*, *EG*: let the common square of *EG* be taken away; therefore what remains, the rectangle contained

Book II.  tained by BE, EF is equal to the square of EH : but the rectangle contained by BE, EF is BD, for EF is equal to ED : therefore the parallelogram BD is equal to the square of HE ; but BD is equal to the rectilineal *figure* A : therefore the rectilineal *figure* A is equal to the square described upon EH.

Wherefore a square is made equal to the given rectilineal *figure* A, the square described upon EH. Which was to be done.

T H E
E L E M E N T S
O F
E U C L I D.
B O O K III.

D E F I N I T I O N S.

1. **E**QUAL circles, are those of which the diameters are ^{Book III.} equal; or of which the *straight lines* from the centers are equal.
2. A straight line is said to touch a circle, which meeting the circle, and being produced does not cut the circle.
3. Circles are said to touch one another, which, meeting each other, do not cut one another.
4. In a circle straight lines are said to be equally distant from the center, when the perpendiculars drawn from the center upon them are equal. 5. But *that line* is said to be more distant upon which the greater perpendicular falls.
6. A segment of a circle is the figure bounded by a straight line and the circumference of a circle. 7. But an angle of a segment is that contained by a straight line and the circumference of a circle.
- VOL. I. I 8. But

Book III.

8. But an angle in a segment is ; when any point is taken in the circumference of the segment, and from it straight lines are joined to the extremities of the straight line which is the base of the segment ; the angle contained by the straight lines so joined. 9. But when the straight lines containing the angle receive any circumference, the angle is said to stand upon that. 10. And a sector of a circle is, when an angle stands at the center of the circle, the figure bounded by the straight lines containing the angle, and the circumference intercepted by them.

11. Similar segments of a circle are such as receive equal angles ; or in which the angles are equal to one another.

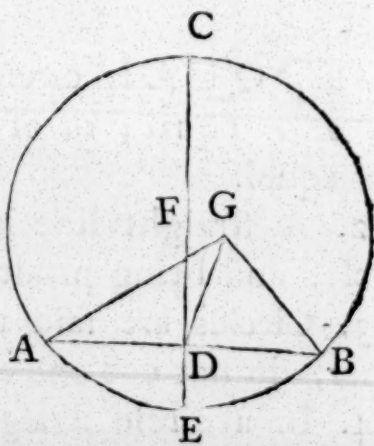
PROP. I.

To find the center of a given circle.

Let the given circle be *the circle ABC* : it is required to find the center of the circle.

Let any straight line AB be drawn in it as it may happen ; and let it be cut in halves at the *point D*, and from the point D let DC be drawn at right angles to AB, and let it be produced to E ; and let CE be cut in halves at the *point F* : I say that F is the center of the circle ABC.

For if not, but if possible let it be G ; and let GA, GD, GB be joined ; and because AD is (by const.) equal to DB, and DG common ; certainly the two AD, DG are equal to the two GD, DB each to each ; and the base GA is (by def. 15.1.) equal to the base GB ; for they are from the center G ; wherefore the angle ADG is equal to the angle GDB ; but when a straight line standing upon a straight line makes the adjacent angles equal to one another, each of the equal angles is a right angle (by def. 10.1.) ; therefore GDB is a right angle ; but FDB is also a right angle ; wherefore FDB is equal to GDB, the greater to the less, which is impossible ; therefore the *point G* is not the center of



of the circle ABC : certainly in the same manner we shall demon- Book III.
strate that neither is any other but the *point* F.

Wherefore the point F is the center of the circle. Which was to be done.

Cor. Certainly from this it is manifest, that if in a circle any straight line, cut any straight line in halves and at right angles, the center of the circle is in the cutting *line*.

P R O P. II.

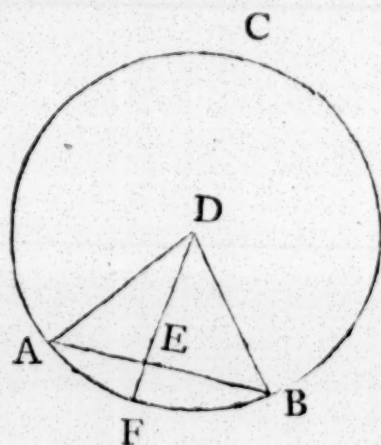
If two accidental points be taken in the circumference of a circle ; the straight line joining these points will fall within the circle.

Let ABC be a circle, and let two accidental points be taken in the circumference of it, *as the points* A, B ; I say that the straight line, joining A and B, will fall within the circle.

For if not, but if possible let AEB fall without ; and let the center of the circle ABC be taken (by 1. 3.), and let it be D ; and let AD, DB be joined ; and let DF be produced to E.

And since DA is equal to DB, therefore the angle DAE (by 5. 1.) is equal to DBE ; and because one side of the triangle DAE is produced the side AEB, therefore (by 16. 1.) the angle DEB is greater than DAE : but DAE is equal to DBE ; therefore DEB is greater than DBE ; but (by 19. 1.) the greater side is extended under the greater angle ; therefore DB is greater than DE ; but DB is equal to DF ; therefore DF is greater than DE ; the less than the greater, *supposing E without the circle*, which is impossible : therefore the straight line joining A and B will not fall without the circle : certainly in the same manner we shall demonstrate that neither will it fall in the circumference : therefore within *the circle*.

Wherefore if two accidental points be taken in the circumference of a circle, the straight line, joining these points, will fall within the circle. Which was to be demonstrated.



Book III.

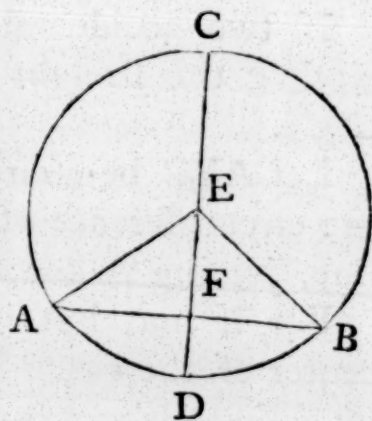
PROP. III.

If in a circle any straight line through the center cut in halves any straight line not through the center, it will also cut it at right angles; and if it cut it at right angles it will also cut it in halves.

Let ABC be a circle, and let CD any straight line in it through the center cut in halves AB any straight line, not through the center, in the point F; I say that it also cuts it at right angles.

For let the center of the circle ABC be taken (by 1. 3.) ; and let it be E, and let EA, EB be joined.

And because AF is equal to FB, and FE common; certainly the two are equal to the two; and the base EA is equal to the base EB; and (by 8. 1.) the angle AFE is equal to the angle BFE: but when a straight line standing upon a straight line makes the adjacent angles equal to one another, each of the equal angles is a right angle; therefore each of the angles AFE, BFE is a right angle: therefore CD through the center cutting in halves AB not passing through the center, also cuts it at right angles.



But let CD cut AB at right angles; I say that it also cuts it in halves; that is, that AF is equal to FB.

For the same things being constructed, because EA from the center is equal to EB, the angle also EAF is equal (by 5. 1.) to EBF; but the right angle AFE is equal to the right angle BFE; therefore there are two triangles EAF, EBF having the two angles equal to the two angles, and one side equal to one side, viz. EF common to them, extended under one of the equal angles; they will therefore also have (by 26. 1.) the remaining sides equal to the remaining sides; wherefore AF is equal to BF.

Wherefore if in a circle, any straight line through the center cut in halves any straight line not through the center; it will also cut it at right angles; and if it cut it at right angles it will also cut it in halves. Which was to be demonstrated.

PROP.

P R O P. IV.

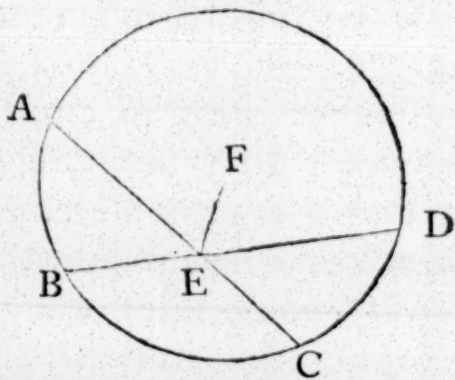
Book III.

If in a circle two straight lines cut one another, not passing through the center; they do not cut one another in halves.

Let ABCD be a circle, and let two straight lines in it AC, BD cut one another in the point E, not passing through the center; I say that they do not cut one another in halves.

For if *it be* possible, let them cut one another in halves, so that AE is equal to EC and BE to ED; and let the center of the circle ABCD be taken (by 1. 3.), and let it be F; and let FE be joined.

Wherefore since a certain straight line FE through the center cuts in halves a certain straight line AC not drawn through the center, it will also cut it (by 3. 3.) at right angles; therefore FEA is a right angle: Again because a certain straight line FE cuts in halves a certain straight line BD, not *passing* through the center, it will also (by 3. 3.) cut it at right angles; wherefore FEB is a right angle; but FEA has been also demonstrated to be a right angle: wherefore the *angle* FEA is equal to the *angle* FEB; the less to the greater which is impossible: wherefore the straight lines AC, BD do not cut one another in halves.



Wherefore if in a circle two straight lines cut one another, not passing through the center; they do not cut one another in halves. Which was to be demonstrated.

P R O P. V.

If two circles cut one another, they will not have the same center.

For let two circles ABC, CDG cut one another in the points, B, C; I say they will not have the same center.

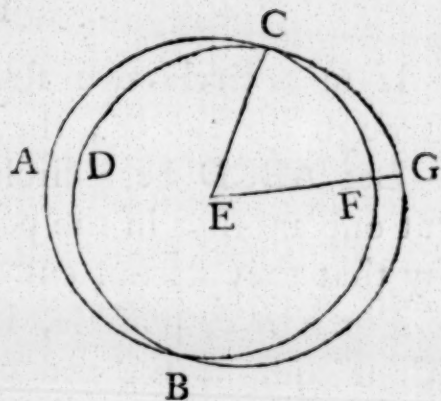
For if *it be* possible, let it be E; and let EC be joined; and let EFG be drawn as it may happen.

And because the point E is the center of the circle ABC, EC is equal to EF: again because the point E is the center of the circle CDG, EC is equal to EG; but EC has been already demonstrated

to

Book III. *to be* equal to EF: wherefore EF is equal to EG, the less to the greater, which is impossible: wherefore the point E is not the center of the circles ABC, CDG.

Therefore if two circles cut one another, they will not have the same center. Which was to be demonstrated.



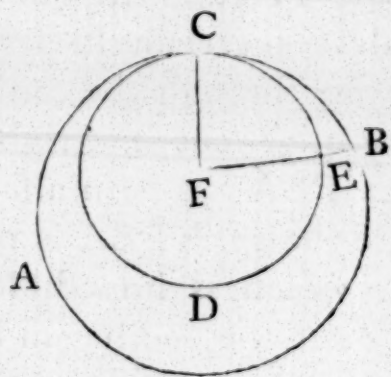
PROP. VI.

If two circles touch one another within, they will not have the same center.

For let the two circles ABC, CDE touch one another in the point C; I say that they will not have the same center.

For if *it be* possible, let it be F, and join FC, and let FEB be drawn as it may happen.

Wherefore since the point F is the center of the circle ABC; FC is equal to FB; again, because the point F is the center of the circle CDE; FC is equal to FE; but FC has been also demonstrated *to be* equal to FB; therefore FE is equal to FB, the less to the greater, which is impossible: wherefore the point F is not the center of the circles ABC, CDE.



Wherefore if two circles touch one another within, they will not have the same center. Which was to be demonstrated.

PROP. VII.

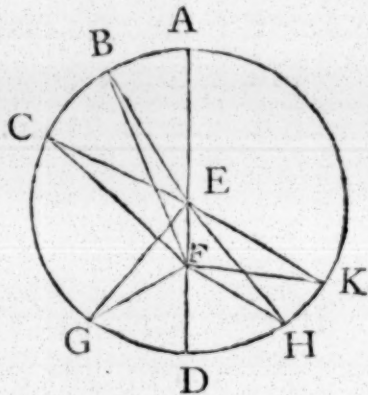
If any point be taken in the diameter of a circle, which is not the center of the circle; and from this point let certain straight lines fall upon the circle, that will be the greatest in which the center *is*; and the remainder *of the diameter* the least; but of the others, the nearer to the line through the center is always greater than one more remote; and two equal straight lines only will fall from the same point upon the circle, on each side of the least.

Let

Let ABCD be a circle, and let AD be its diameter; and let any ^{Book III.} point F be taken in AD, which is not the center of the circle; and let E be the center of the circle; and from the point F let certain straight lines FB, FC, FG fall upon the circle ABCD: I say that FA is the greatest, and FD the least; but of the others FB is greater than FC; and FC than FG.

For let BE, CE, GE be joined.

And because the two sides of every triangle are greater than the remaining *side*; therefore EB, EF are greater than BF; but AE is equal to BE; wherefore BE, EF are equal to AF: therefore AF is greater than BF: Again, because BE is equal to CE, and FE common; certainly the two BE, EF are equal to the two CE, EF; but the angle BEF is also greater than the angle CEF; wherefore (by 24. 1.) the base BF is greater than the base CF. Certainly for the same reason also CF is greater than FG.



Again, since GF, FE are (by 20. 1.) greater than EG; and EG is equal to ED; therefore GF, FE are greater than ED; Let EF *which is common* be taken away; therefore the remainder GF is greater than the remainder FD; wherefore FA is the greatest, and FD the least; and FB is greater than FC; and FC than FG.

I say that also only two equal straight lines will fall from the point F upon the circle ABCD, *one* on each *side* of the least FD: for let the *angle* FEH be made with the straight line EF, and at the point E in it, equal to the angle GEF; and let FH be joined wherefore because GE is equal to EH, and EF common; certainly the two GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; wherefore (by 4. 1.) the base FG is equal to the base FH: I say that another *line* equal to FG will not fall upon the circle, from the point F: for if it be possible, let FK fall; and since FK is equal to FG, but FH is equal to FG; therefore FK is equal FH; the nearer to the *line* through the center equal to the *one* more remote; which is impossible.

OR

Book III. *OR THUS.* Let EK be joined; and because GE is equal to EK, and FE common, and the base GF equal to the base FK; therefore (by 8. 1.) the angle GEF is equal to the angle KEF; but the angle GEF is equal to the *angle* HEF; and therefore the *angle* HEF is equal to the *angle* KEF; the less to the greater which is impossible: therefore no other line will fall from the point F upon the circle equal to GF: therefore one only.

Wherefore if any point be taken in the diameter of a circle &c. Which was to be demonstrated.

P R O P. VIII.

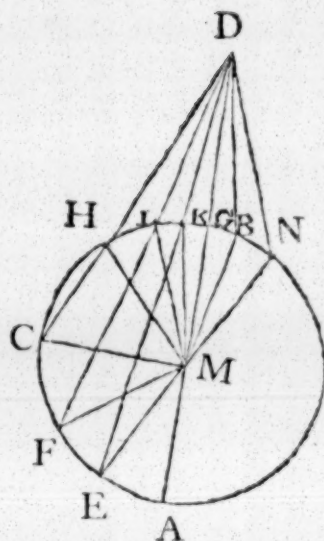
If any point be taken without a circle, and certain straight lines be drawn from the point to the circle, one of which *passes* through the center, but the other *lines* as it may happen; of those straight lines falling upon the concave circumference, the *one passing* through the center is the greatest: but of the other *lines*, the *one* nearer the *line passing* through the center will always be greater than that more remote: But of the straight lines falling upon the convex circumference, the least is that between the point and the diameter: but of the other lines the nearer to the least is less than that more remote; and two equal straight lines only will fall upon the circle from the point, *one* on each *side* of the least.

Let ABC be a circle; and let any point D be taken without the circle ABC: and let certain straight lines be drawn from it to the circle viz. DA, DE, DF, DC; and let AD be through the center; I say that of the straight lines falling upon AEFC the concave circumference, the greatest is DA the *line* through the center: but the *line* nearer to the *one* through the center will be greater than *one* more remote; viz DE than DF and DF *greater* than DC; but of the straight lines falling upon the convex circumference HLKG, DG is the least, which is between the point D and the diameter AG; but the nearer to the least DG is less than one more remote; viz. DK *less* than DL and DL than DH.

For let the center of the circle ABC be taken; and let it be M; and let ME, MF, MC, MK, ML, MH be joined.

And

And because AM is equal to EM; let MD *which is common* be added; therefore AD is equal to EM, MD: but (by 20. 1.) EM, MD are greater than ED; and therefore AD is greater than ED: Again because ME is equal to MF; and MD common; therefore EM, MD are equal to MF, MD; and the angle EMD is greater than the angle FMD; wherefore (by 24. 1.) the base ED is greater than the base FD: certainly in the same manner we shall demonstrate also that DF is greater than CD: wherefore AD is the greatest; and DE greater than DF; and DF than DC.



And since MK, KD are greater than MD; and MG equal to MK, therefore the remainder KD is greater than the remainder GD; so that GD is less than KD, wherefore it is the least. And because upon MD one of the sides of the triangle MLD two straight lines MK, KD are joined together within the triangle; therefore (by 21. 1.) MK, KD are less than ML, LD; of which MK is equal to ML; therefore the remainder DK is less than the remainder DL: certainly in the same manner we shall demonstrate that DL is less than DH: wherefore DG is the least; and DK is less than DL; and DL than DH.

Also I say that only two equal straight lines will fall from the point D upon the circle; *one* on each side of DG the least: with the straight line MD, and at the point M in it, let the angle DMB be made equal to the angle KMD; and let DB be joined; and because MK is equal to MB, and MD common; certainly the two KM, MD are equal to the two BM, MD, each to each; and the angle KMD is equal to the angle BMD: wherefore (by 4. 1.) the base DK is equal to the base DB: I say that another *straight line* equal to the straight line DK will not fall upon the circle from the point D: for if it be possible let it fall; and let it be DN; because therefore DK is equal to DN; but DK is equal to DB; therefore also DB is equal to DN; the nearer to the least DG equal to *one* more remote, which has been shewn to be impossible.

OR OTHERWISE. Let MN be joined, and because KM is equal

Book III. to MN, and MD common, also the base DK equal to the base DN; wherefore the angle KMD is equal to the angle DMN; but the *angle* KMD is equal to BMD: therefore also BMD is equal to NMD; the less to the greater, which is impossible: Therefore more than two straight lines will not fall upon the circle ABC from the point D, *one* on each *side* of the least GD.

Wherefore if any point be taken without a circle &c. Which was to be demonstrated.

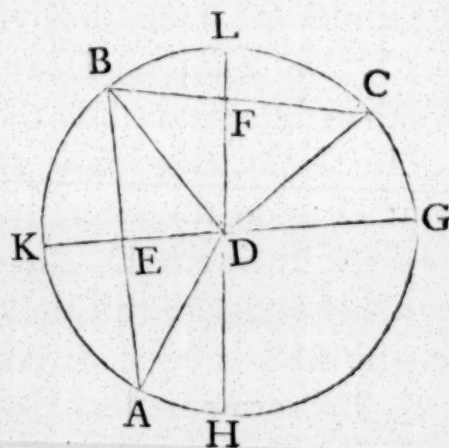
PROP. IX.

If any point be taken within a circle, and more than two equal straight lines fall from the point upon the circle: the point taken is the center of the circle.

Let ABC be a circle, and the point D within it; and let DA, DB, DC; more than two equal straight lines, fall from the *point* D to the circle ABC: I say that the point D is the center of the circle ABC.

For let AB, BC be joined, and let them be cut in halves in the points E, F; ED, DF being joined; let them be produced to the points G, K, H, L.

Wherefore because AE is equal to EB, and ED common: certainly the two AE, ED are equal to the two BE, ED; and the base DA is equal to the base DB; therefore (by 8. 1.) the angle AED is equal to the angle BED; wherefore each of the angles AED, BED is a right angle; wherefore GK cutting AB in halves cuts it also at right angles; and because (by cor. prop. 1. 3.) if in a circle any straight line cut any



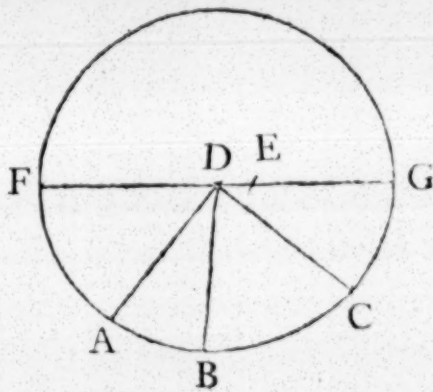
straight line in halves and at right angles, the center of the circle is in the cutting line; therefore the center of the circle ABC is in GK: certainly for the same reason also the center of the circle ABC is in HL; and the straight lines GK, HL have no other *point* in common, but the point D; therefore the point D is the center of the circle ABC.

Where-

Wherefore if any point be taken within a circle; and more than two equal straight lines fall from the point upon the circle; the point taken is the center of the circle. Which was to be demonstrated. Book III.

OTHERWISE. For let there be, any point D, taken within the circle ABC and let more than two equal straight lines DA, DB, DC fall from the point D upon the circle ABC; I say that D the point taken is the center of the circle ABC.

For if not; but if possible let it be E; and DE being joined, let it be produced to the points F, G; but FG is a diameter of the circle ABC: therefore because a certain point hath been taken in FG the diameter of the circle ABC, which is not the center of the circle, viz. the point D; DG will be the greatest (by prop. 7.3.) and DC greater than DB; and DB than DA; but they are also equal; which is impossible: wherefore E is not the center of the circle ABC: certainly in the same manner we shall demonstrate, that neither is any other but the point D: wherefore the point D is the center of the circle ABC.

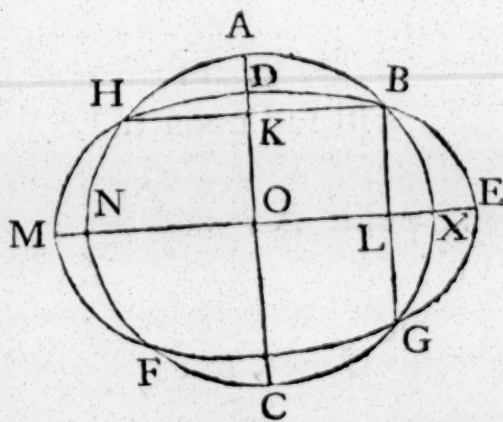


P R O P. X.

A circle does not cut a circle in more points than two.

For if it be possible; let the circle ABC cut the circle DEF in more points than two, viz. in B, G, H; and BG, BH being joined, let them be cut in halves in the points K, L; and from the points K, L having drawn KC, LM at right angles to BG, BH let them be produced to the points A, E.

Wherefore because in the circle ABC a certain straight line AC cuts in halves a certain straight line BH and at right angles; therefore the center of the circle ABC is in the straight line AC: again, because in the same circle ABC a certain straight line NX cuts in halves and at right angles a certain straight line BG;



K 2

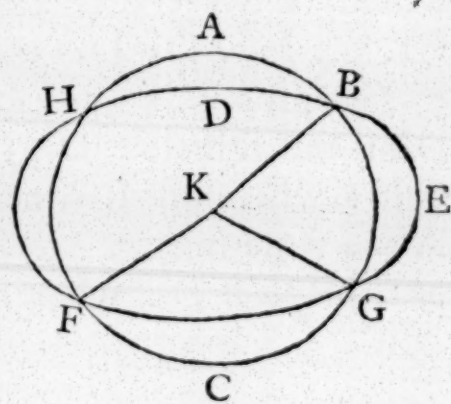
there-

Book III. therefore the center of the circle ABC is in NX ; but it has also been demonstrated to be in AC ; and the straight lines AC, NX meet one another in no *point* but the *point* O ; therefore the point O is the center of the circle ABC : certainly in the same manner we shall demonstrate that O is the center of the circle DEF ; therefore two circles ABC, DEF cutting one another have the same center the *point* O which is impossible (by 5. 3.).

Wherefore a circle does not cut a circle in more points than two. Which was to be demonstrated.

OTHERWISE. For again, let the circle ABC cut the circle DEF in more points than two ; viz. in the *points* B, G, F ; and let the center of the circle ABC be taken the *point* K ; and let KF, KG, KB be joined.

Wherefore since a certain point hath been taken within the circle DEF, the point K ; and from K more than two equal straight lines KB, KF, KG have fallen upon the circle DEF : therefore (by 9. 3.) the point K is the center of the circle DEF ; but K is also the center of the circle ABC ; therefore two circles cutting one another have the same center K ; which (by 5. 3.) is impossible.



Wherefore a circle does not cut a circle in more points than two. Which was to be demonstrated.

PROP. XI.

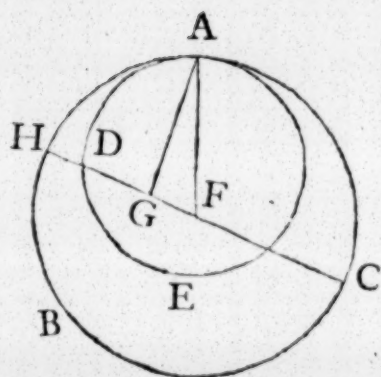
If two circles touch one another inwardly, and their centers be taken ; the straight line joining the centers of them, being produced, will fall upon the contact of the circles.

For let two circles ABC, ADE touch one another inwardly at the point A ; and let F the center of the circle ABC be taken, and G the center of the circle ADE : I say that the straight line joining G and F being produced will fall upon the point A.

For if not, but if possible let it fall as the *line* FGDH ; and let AF, AG be joined.

Where-

Wherefore because AG , GF are greater than FA , that is than FH ; for FA is equal to FH for they are both from the center; let FG which is common be taken away; therefore the remainder AG is greater than the remainder GH ; but AG is equal to GD (by def. 15. 1.); therefore GD is greater than GH ; the less than the greater; which is impossible: wherefore the straight line joining F and G will not fall without the contact at A ; therefore upon it.



Wherefore if two circles touch one another inwardly, and their centers be taken; the straight line joining the centers of them, being produced, will fall upon the contact of the circles. Which was to be demonstrated.

OTHERWISE. But let it fall as GFC ; and let GFC be produced in a straight line to the point H ; and let AG , AF be joined.

Wherefore since AG , GF are greater (by 20. 1.) than AF ; but AF is equal to CF : that is to FH ; let FG *which is* common be taken away; therefore the remainder AG is greater than the remainder GH ; that is GD *is greater* than GH ; the less than the greater which is impossible; in the same manner we shall shew the same *thing* to be absurd, if the center of the greater circle be without the less.

P R O P. XII.

If two circles touch one another outwardly, the straight line joining their centers will pass through the contact.

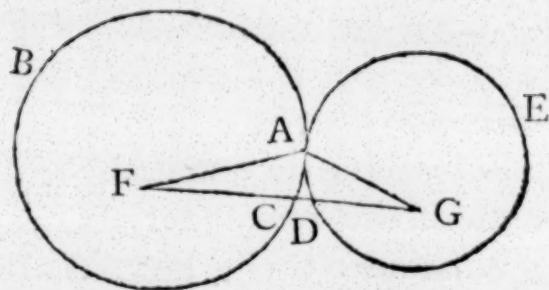
For let two circles, ABC , ADE touch one another outwardly in the point A ; and let F the center of the circle ABC be taken; and G the *center* of ADE ; I say that the straight line joining F and G will pass through the contact A .

For if not, but if possible let it pass as $FCDG$; and let AF , AG be joined.

Where-

Book III.

Wherefore because the point F is the center of the circle ABC ; FA is equal to FC : again because the point G is the center of the circle ADE ; AG is equal to GD ; but FA has been demonstrated to be equal to FC ; therefore FA, AG are equal to FC, DG ; so that the whole FG is greater than FA, AG ; but *it is also less* (by 20. 1.) ; which is impossible: wherefore the straight line joining F and G will not not pass through the contact at A ; therefore *it will pass* through it.



Wherefore if two circles touch one another outwardly, the straight line joining their centers will pass through the contact: Which was to be demonstrated.

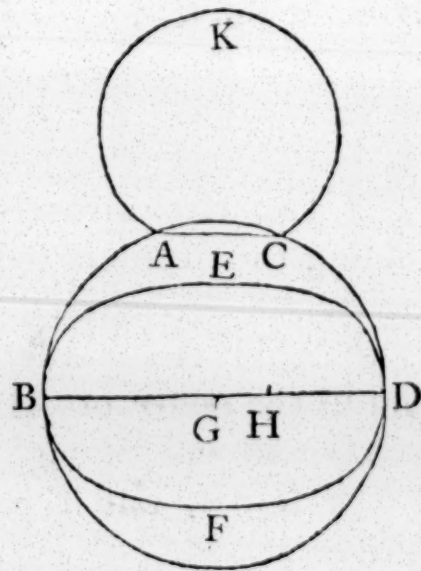
P R O P. XIII.

A circle will not touch a circle in more points than one, whether it touch it within or without.

For if it be possible, let the circle $ABDC$ first touch the circle $EBFD$ within, in more points than one; viz. in the points, B, D .

And let G , the center of the circle $ABDC$ be taken (by 1. 3.); and H the center of the circle $EBFD$.

Wherefore the straight line joining G and H will (by 11. 3.) fall upon the points B, D : let it fall as $BGHD$; and since the point G is the center of the circle $ABDC$; BG is equal to GD ; wherefore BG is greater than HD ; therefore BH is much greater than HD ; Again because the point H is the center of the circle $EBFD$; BH is equal to HD ; but it hath also been demonstrated to be much greater than it; which is impossible; wherefore a circle does not touch a circle in more points than one inwardly.



I say

I say that neither does it outwardly. For if it be possible; let Book III.
the circle ACK touch the circle ABDC without in more points {
than one, in the points A, C and let AC be joined.

Wherefore because any two points A, C have been taken in the circumference of each of the circles ABDC, ACK; the straight line joining these points will fall (by 2. 3.) within both: but it hath fallen within the circle ABDC, and without the circle ACK; which is absurd: wherefore a circle does not touch a circle outwardly in more points than one; but it has been demonstrated that neither *does it* inwardly.

Wherefore a circle does not touch a circle in more points than one, whether it touch it on the inside or on the outside. Which was to be demonstrated.

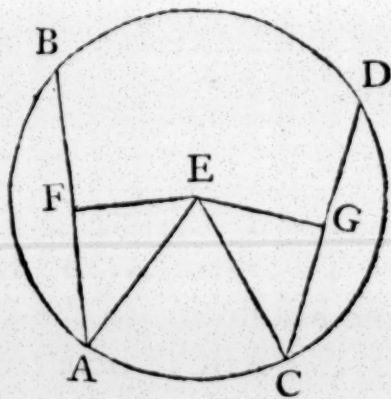
P R O P. XIV.

The equal straight lines, in a circle, are equally distant from the center; and the straight lines being equally distant from the center are equal to one another.

Let ABDC be a circle; and let AB, CD be equal straight lines in it; I say that they are equally distant from the center.

For let the center of the circle ABDC be taken (by 1. 3.); and let it be E; and let EF, EG be drawn perpendiculars from E to AB, CD; and let AE, EC be joined.

Wherefore since EF, a certain straight line through the center, cuts a certain straight line AB, not through the center, at right angles; it will cut it also in halves (by 3. 3.); wherefore AF is equal to FB: therefore AB is the double of AF. Certainly, for the same reason also, CD is double of CG; and AB is equal to CD; wherefore AF is equal to CG (by com. not. 7.): and because AE is equal to EC; also the *square* of AE is equal to the *square* of EC: but the *squares* of AF, FE are equal (by 47. 1.) to the *square* of AE; for the angle at F is a right angle: and the *squares* of EG, GC are equal to the *square* of EC; for the
angle



Book III. angle at G is a right angle : wherefore the *squares* of AF, FE are equal to the *squares* of CG, GE ; of which the *square* of AF is equal to the *square* of CG ; for AF is equal to CG ; therefore the remainder the *square* of FE is equal to the remainder the *square* of EG ; wherefore FE is equal to EG : but in a circle, straight lines are said to be equally distant from the center, when perpendiculars drawn from the center upon them are equal (by def. 4. 3.) therefore AB, CD are equally distant from the center

But let the straight lines AB, CD be equally distant from the center, that is, let FE be equal to EG : I say that AB is also equal to CD.

For the same things being constructed ; certainly we shall demonstrate, in the same manner, that AB is the double of AF ; and CD of CG : and because AE is equal to EC ; the *square* of AE is also equal to the *square* of EC ; but the *squares* of EF, FA are equal to the *square* of AE ; and the *squares* of EG, GC are equal to the *square* of EC ; therefore the *squares* of EF, FA are equal to the *squares* of EG, GC ; of which the *square* of EG is equal to the *square* of EF ; therefore the remaining *square* of AF is equal to the remaining *square* of CG ; wherefore AF is equal to CG ; also AB is the double of AF, and CD the double of CG ; wherefore AB is equal to CD.

Wherefore the equal straight lines, in a circle, are equally distant from the center ; and the straight lines, being equally distant from the center, are equal to one another. Which was to be demonstrated.

P R O P. XV.

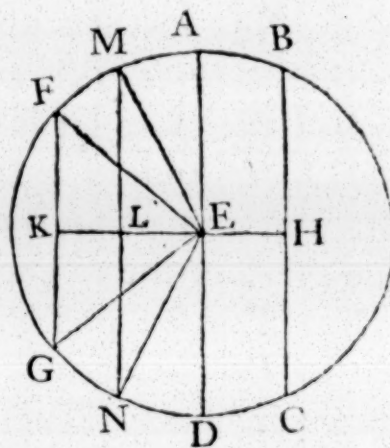
The diameter is the greatest *line* in a circle ; but of the others, the *one* nearer to the center is greater than *any one* more remote.

Let ABCD be a circle ; and let AD be a diameter of it ; and E the center ; and let BC be nearer the center E ; but FG more remote ; I say that AD is the greatest, and BC greater than FG.

For let EH, EK be drawn perpendiculars from the center, to BC, FG ; and because BC is nearer to the center, but FG more remote ; therefore (by def. 5. 3.) EK is greater than EH ; let EL
be

be made equal to EH; and through the point L, LM being drawn Book III.
at right angles to EK let it be produced to N; and let EM, EN, {
EF, EG be joined.

And because EH is equal to EL ; (by 14. 3.) BC it equal to MN : Again since AE is equal to EM and ED to EN ; therefore AD is equal to ME, EN ; but (by 20. 1.) ME, EN are greater than MN ; therefore also AD is greater than MN : but MN is equal to BC ; therefore AD is greater than BC : And because the two ME, EN are equal to the two FE, EG and the angle MEN is greater than the angle FEG ; therefore (by 24. 1.) the base MN is greater than the base FG ; but MN has been demonstrated *to be* equal to BC ; and BC is also greater than FG : wherefore the diameter AD *is* the greatest ; and BC greater than FG.



Wherefore the diameter is the greatest *line* in a circle ; but of the others the *one* nearer to the center is greater than *any one* more remote. Which was to be demonſtrated.

P R O P. XVI.

The *straight line* drawn at right angles to the diameter of a circle, from the extremity, will fall without the circle; and into the place between the straight line and the circumference another straight line will not fall; and the angle of the semicircle is greater than every acute rectilineal angle: but the remainder less.

Let ABC be a circle about the center D, and AB its diameter ; I say that the *straight line* drawn from its extremity, from the point A, at right angles to AB, will fall without the circle.

For if not, but if possible let it fall within, as AC and let DC be joined.

And because DA is equal to DC, the angle DAC (by 5. 1.) is also equal to the angle ACD; but DAC is a right angle (by const.) wherefore also ACD is a right angle, therefore the angles DAC, ACD are equal to two right angles; which (by 17. 1.) is impos-

Book III. fible ; wherefore the *straight line* drawn from the point A, at right angles to AB will not fall within the circle : certainly in like manner we shall demonstrate that neither *will it fall* upon the circumference ; therefore let it fall without as AE.

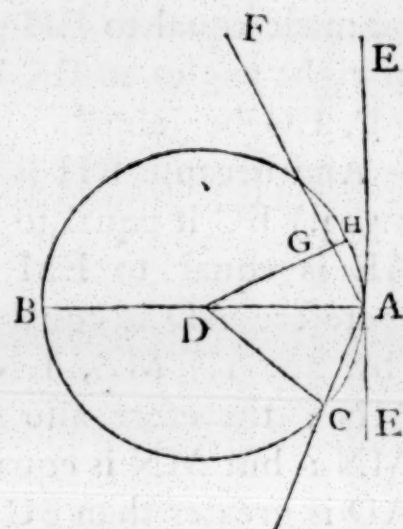
I say that another straight line will not fall into the place, between the straight line AE and the circumference HAC.

For if it be possible let it fall as FA ; and let DG be drawn, from the point D, perpendicular to FA.

And because AGD is a right angle, but DAG (by 17. 1.) less than a right angle ; therefore (by 19. 1.) AD is greater than DG ; but AD is equal to DH ; therefore DH is greater than DG ; the less than the greater, *supposing G without the circle*, which is impossible ; wherefore another straight line will not fall into the place between the straight line and the circumference.

Also I say that the angle of the semicircle ; the *angle* contained by the straight line BA and the circumference HA ; is greater than every acute rectilineal angle ; but the remainder, the *angle* contained by the circumference HA and the straight line AE is less than every acute rectilineal angle.

For if there be any rectilineal angle, greater than that contained by the straight line BA and the circumference HA ; but less than that contained by the circumference HA and the straight line AE ; a straight line will fall into the place between the circumference HA, and the straight line AE ; which will make *an angle* contained by straight lines greater, than that contained by the straight line BA and the circumference HA ; but less than that contained by the circumference HA and the straight line AE : but it does not fall : therefore, the acute angle contained by the straight lines will not be greater than the angle contained by the straight line BA and the circumference HA, nor less than the *angle* contained by the circumference HA and the straight line AE. Which was to be demonstrated.



Cor.

Cor. From these *demonstrations* it is manifest that the *straight line* Book III. drawn from the extremity at right angles to the diameter of the circle, touches the circle: and that a straight line touches a circle in one point only: because (by 2. 3.) the straight line meeting it in two *points* has been shewn to fall within it.

P R O P. XVII.

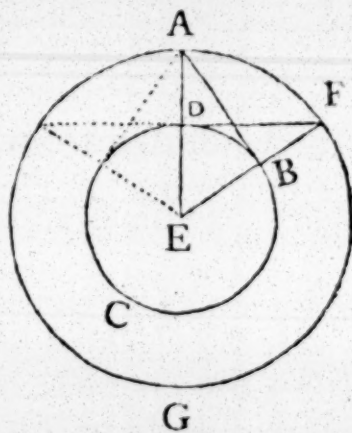
From a given point to draw a straight line touching a given circle.

Let A be the given point, and BCD the given circle; it is required from the point A to draw a straight line touching the circle BCD.

For let E the center of the circle be taken (by 1. 3.); and let AE be joined; and with the center E and at the distance EA let the circle AFG be described; and from the point D, let DF be drawn at right angles to AE; and let EBF, AB be joined: I say that from the point A the *straight line* AB hath been drawn touching the circle BCD.

For since E is the center of the circles BCD, AFG; therefore EA is equal to EF; and ED to EB: certainly the two AE, EB are equal to the two FE, ED; and they contain a common angle, the *angle* at E; wherefore (by 4. 1.) the base DF is equal to the base AB and the triangle DEF is equal to the triangle EBA; and the remaining angles to the remaining angles; therefore the *angle* EBA is equal to EDF; but EDF is a right angle; wherefore EBA is a right angle; and EB is from the center: but the *straight line* drawn from the extremity at right angles to the diameter of the circle touches the circle: wherefore AB touches the circle.

Wherefore from a given point, the *point* A, a straight line AB hath been drawn touching the circle BCD. Which was to be done.



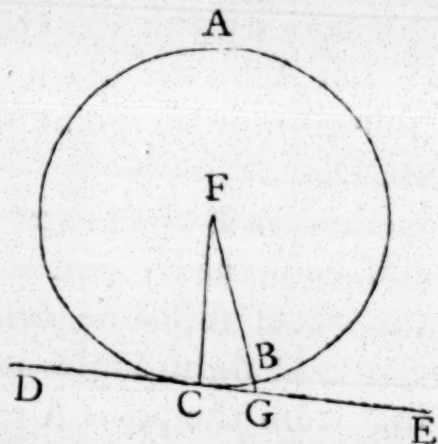
PROP. XVIII.

If any straight line touch a circle, and any straight line be drawn from the center to the *point of contact*; the *line* drawn will be perpendicular to the touching *line*.

Let any straight line DE touch the circle ABC, in the point C; and let F the center of the circle ABC be taken; and from F to C let FC be drawn; I say that FC is perpendicular to DE.

For if not, let FG be drawn from the point F perpendicular to DE.

Wherefore because FGC is a right angle, therefore GCF is an acute angle; but the greater side is extended under the greater angle; therefore FC is greater than FG; but FC is equal to FB: wherefore FB is greater than FG; the less than the greater which is impossible: therefore FG is not perpendicular to DE: certainly in the same manner we shall demonstrate, that neither is any other but FC: wherefore FC is perpendicular to DE.



Wherefore if any straight line touch a circle; and any straight line be drawn from the center to the contact; the *line* drawn is perpendicular to the *tangent*. Which was to be demonstrated.

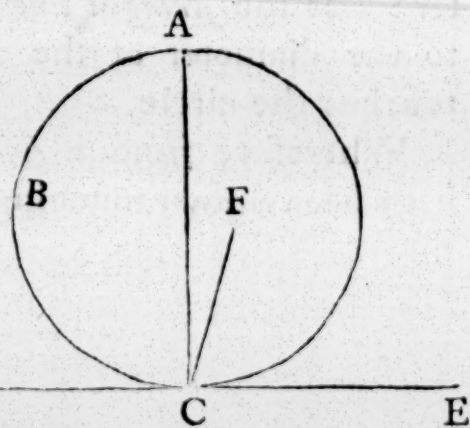
PROP. XIX.

If any straight line touch a circle, and a straight line be drawn, from the contact, at right angles to the touching *line*; the center of the circle will be in the *line* so drawn.

For let any straight line DE touch the circle ABC in the point C; and from C let CA be drawn at right angles to DE; I say that the center of the circle is in AC.

For if not, but if possible let it be F: and let CF be joined.

Wherefore because a certain straight line DE touches the circle ABC; and FC hath been drawn from the center to the contact; therefore FC is perpendicular to DE (by 18.3.); therefore FCE is a right angle; but (by D



const.) ACE is a right angle ; wherefore FCE is equal to ACE ; *Book III.*
the less to the greater ; which is impossible ; wherefore F is not
the center of the circle ABC : certainly in the same manner we
shall demonstrate, that neither is it any other, but *some point* in AC.

Wherefore if any straight line touch a circle, and a straight line
be drawn, from the contact, at right angles to the touching line ;
the center of the circle will be in the line so drawn. Which was
to be demonstrated.

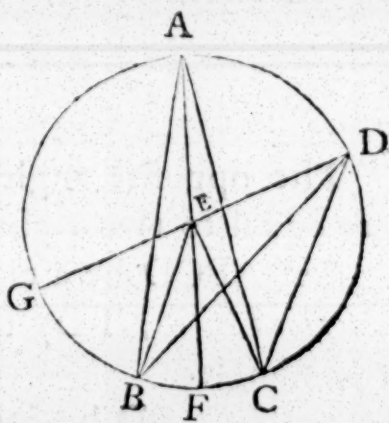
PROP. XX.

In a circle, the angle at the center is double of the angle at
the circumference ; when the angles have the same circumference
for a base.

Let ABC be a circle, and let BEC be an angle at the center of
it ; but BAC at the circumference *of it* ; and let them have the
same circumference BC *for a base* ; I say that the angle BEC is
double of the angle BAC.

For AE being joined let it be produced
to F.

Wherefore because EA is equal to EB ;
also (by 5. 1.) the angle EAB is equal to
EBA ; wherefore the angles EAB, EBA
are double of EAB : but (by 32. 1.) BEF
is equal to EAB, EBA ; wherefore BEF is
double of EAB : Certainly for the same
reason FEC is double of EAC : therefore
the whole BEC is double of the whole BAC.



Let it be bent in a different direction ; and let the other angle
be BDC ; and DE being joined let it be produced to G : certainly
in the same manner we shall demonstrate that the angle GEC is
double of the *angle* GDC ; of which GEB is double of GDB ;
therefore the remainder BEC is double of the remainder BDC.

Wherefore in a circle, the angle at the center is double of the
angle at the circumference ; when the angles have the same cir-
cumference *for a base.* Which was to be demonstrated.

PROP.

Book III.

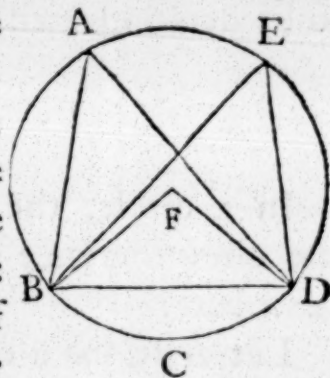
PROP. XXI.

In a circle, the angles in the same segment are equal to one another.

Let $ABCD$ be a circle, and let the angles BAD , BED be in the segment $BAED$; I say that the angles BAD , BED are equal to one another.

For let the center of the circle $ABCD$ be taken; and let it be F ; and join BF , FD .

And because the angle BFD is at the center; but BAD at the circumference, and have the same circumference BCD for a base; therefore the angle BFD is double of the angle BAD : certainly for the same reason BFD is double of BED : wherefore (by com. not. 7.) the angle BAD is equal to BED .



Wherefore in a circle, the angles in the same segment are equal to one another. Which was to be demonstrated.

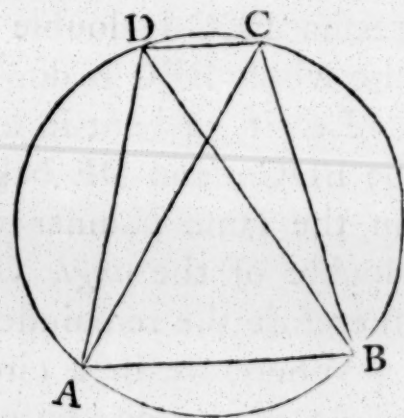
PROP. XXII.

The opposite angles, of quadrilateral figures inscribed in circles, are equal to two right angles.

Let $ABCD$ be a circle; and let $ABCD$ be a quadrilateral figure inscribed in it; I say that the opposite angles of it are equal to two right angles.

Let AC , BD be joined.

And because (by 32. 1.) the three angles of every triangle are equal to two right angles; CAB , ABC , BCA the three angles of the triangle ABC are equal to two right angles: but CAB is equal to BDC (by 21. 3.); for they are in the same segment $BADC$: and ACB is equal to ADB (by 21. 3.): for they are in the same segment $ADCB$; wherefore the whole angle ADC is equal to the two angles BAC , ACB ; let ABC which is common be added; therefore



fore the *three angles* ABC, BAC, ACB are equal to the *two* ABC, ADC : but the angles ABC, BAC, ACB are equal to two right angles; also the angles ABC, ADC are equal to two right angles: Certainly in the same manner we shall demonstrate, that the angles BAD, DCB are equal to two right angles.

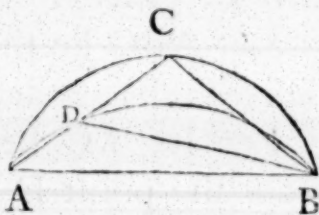
Wherefore the opposite angles, of quadrilateral *figures inscribed* in circles, are equal to two right angles. Which was to be demonstrated.

P R O P. XXIII.

Two similar and unequal segments of circles will not stand upon the same straight line, towards the same parts.

For if it be possible; let ACB, ADB two similar and unequal segments of circles stand upon the same straight line AB , towards the same parts, and let ADC be drawn; and let CB, DB be joined.

Wherefore because the segment ACB is similar to the segment ADB : and similar segments of circles are (by def. 11. 3.) those receiving equal angles; therefore the angle ACB is equal to the angle ADB ; the outward to the inward; which (by 16. 1.) is impossible.



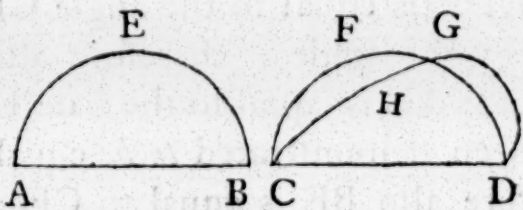
Wherefore two similar and unequal segments of circles will not stand upon the same straight line, towards the same parts. Which was to be demonstrated.

P R O P. XXIV.

The similar segments of circles, upon equal straight lines, are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD ; I say that the segment AEB is equal to the segment CFD .

For the segment AEB being applied to the segment CFD ; and the point A being placed upon the point C ; and the straight line AB upon CD ; the point B will apply



itself to the point D ; because AB is equal to CD ; but the straight

Book III. straight line AB applying itself to CD, the segment AEB (by 23. 3.) will apply itself to CFD : for if the straight line AB shall apply itself to CD ; but the segment AEB will not apply itself to CFD ; but shall change its direction as CHGD : but a circle does not cut a circle in more points than two (by 10. 3.) ; but the *circle* CHGD cuts CFD in more points than two ; viz. in C, G, D ; which is impossible : wherefore the straight line AB being applied to CD, also the segment AEB will not, not apply itself to CFD ; therefore it will apply *itself*, and will be equal to it.

Wherefore the similar segments of circles, upon equal straight lines, are equal to one another, Which was to be demonstrated.

P R O P. XXV.

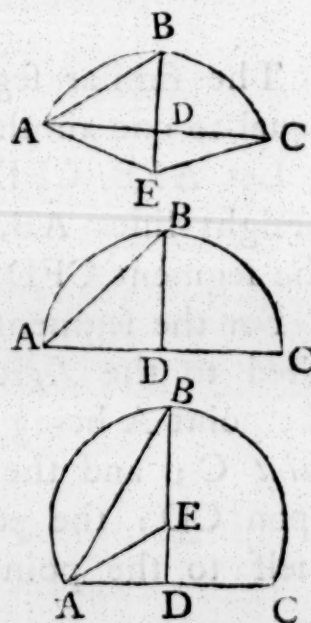
A segment of a circle being given to describe the circle of which it is a segment.

Let ABC be the given segment of the circle ; it is required to describe the circle, of which ABC is a segment.

For let AC be cut in halves (by 10. 1.) in the *point* D ; and from the point D let DB be drawn at right angles to AC ; and let AB be joined : therefore the angle ABD is either greater, equal or less than the *angle* BAD.

First, let it be greater ; and let the *angle* BAE be made, with the straight line AB and at the point A in it, equal to the angle ABD ; and let DB be produced to E ; and let EC be joined.

Wherefore because the angle ABE is equal to BAE ; therefore (by 6. 1.) the straight line BE is equal to EA : And because AD is (by const.) equal to DC and DE common ; certainly the two AD, DE are equal to the two CD, DE each to each ; and the angle ADE is equal to the angle CDE ; for each is a right angle ; therefore also (by 4. 1.) the base AE is equal to the base EC ; but AE has been demonstrated *to be* equal to EB ; therefore also BE is equal to CE ; wherefore the three *straight lines* AE, EB, EC are equal to one another : therefore a circle described with



the center E and at the distance of one of the *lines* AE, EB, EC Book III. will also pass through the remaining points, and will be the *circle* to be described (by 3. 3.) : Wherefore the segment of a circle being given the circle has been described ; and it is manifest that the segment ABC is less than a semicircle ; on this account, because E the center of it falls without.

In like manner, if the angle ABD is equal to the angle BAD ; AD becoming equal to either of the *lines* BD, DC ; therefore the three DA, DB, DC will be equal to one another ; and D will be the center of the circle completed ; and certainly ABC will be a semicircle.

But if the *angle* ABD be less than BAD ; also let us make with the straight line AB, and at the point A in it, an angle equal to the *angle* ABD ; the center will fall within the segment ABC in the *straight line* DB ; and certainly the segment ABC will be greater than a semicircle.

Wherefore a segment of a circle being given, the circle has been described, of which it is a segment. Which was to be done.

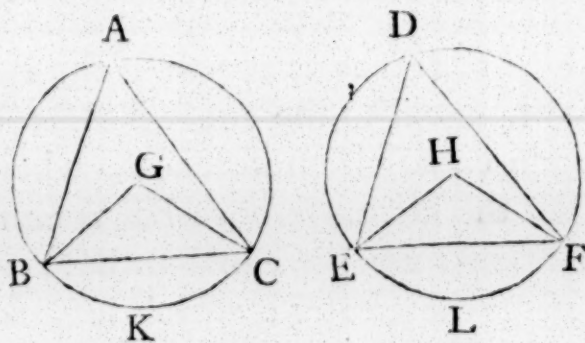
P R O P. XXVI.

In equal circles, the equal angles stand upon equal circumferences ; whether they stand at the center, or at the circumference.

Let ABC, DEF be equal circles ; and let BGC, EHF be equal angles in them, at *their* centers ; and BAC, EDF at the circumference : I say that the circumference BKC is equal to the circumference ELF.

For let BC, EF be joined.

And because the circles ABC, DEF are equal ; the straight lines from their centers are equal (by def. 1. 3.) : the two BG, GC are equal to the two EH, HF and (by supp.) the angle at G is equal to the *angle* at H : wherefore the base BC is equal to the base EF : and because the angle at A is equal to the *angle* at D ; therefore the segment BAC



Book III. is similar to the segment EDF (by def. 11. 3.); and they are upon equal straight lines BC, EF; but similar segments *being* upon equal straight lines are equal (by 24. 3.); wherefore the segment BAC is equal to the segment EDF; but also the whole circle ABC is equal to the whole circle DEF; therefore the remaining segment BKC is equal to the remaining segment ELF; wherefore the circumference BKC is equal to the circumference ELF.

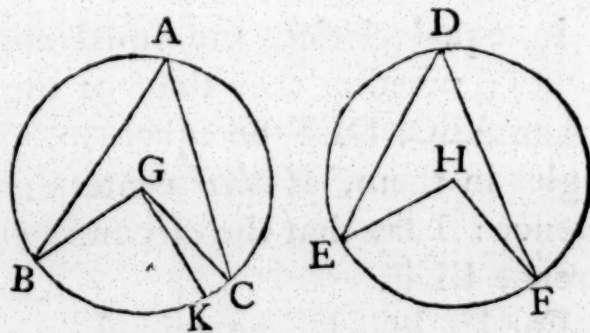
Wherefore in equal circles, the equal angles stand upon equal circumferences; whether they stand at the center or at the circumference. Which was to be demonstrated.

P R O P. XXVII.

In equal circles, the angles which stand upon equal circumferences are equal to one another; whether they stand at the center or at the circumference.

For let the angles BGC, EHF stand at the centers G, H in the equal circles ABC, DEF; and upon the equal circumferences BC, EF; and the angles BAC, EDF at the circumferences; I say that the angle BGC is equal to the angle EHF; and BAC to EDF.

If the angle BGC be equal to EHF; it is plain (by 20. 3.) that the angle BAC is equal to the angle EDF; but if the angle BGC be not equal to the angle EHF; one of them is greater; let BGC be the greater, and let the angle BGK



(by 23. 1.) be made with the straight line BG and at the point G in it, equal to the angle EHF: but (by 26. 3.) equal angles stand upon equal circumferences, when they are at the centers; wherefore the circumference BK is equal to the circumference EF; but EF (by supp.) is equal to BC; wherefore BK is equal to BC; the less to the greater; which is impossible: wherefore the angle BGC is not unequal to EHF; therefore equal; and the angle at A is half of the angle BGC; and the angle at D is half of the angle EHF: wherefore the angle at A is equal to the angle at D.

Where-

Wherefore in equal circles, the angles which stand upon equal cir- Book III.
cumferences are equal to one another ; whether they stand at the ~
center or at the circumference. Which was to be demonstrated.

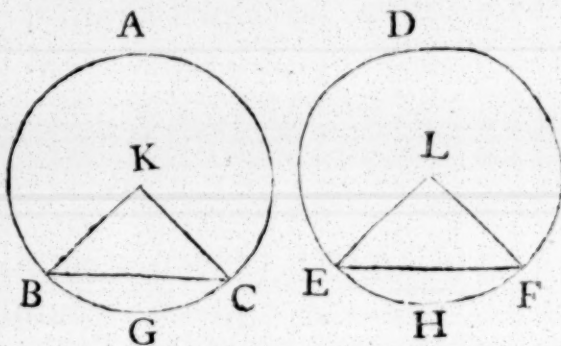
P R O P. XXVIII.

Equal straight lines in equal circles cut off equal circumferences ; the greater *equal* to the greater ; and the less *equal* to the less.

Let ABC, DEF be equal circles ; and let BC, EF be equal straight lines in them, cutting off the greater circumferences BAC, EDF ; and BGC, EHF the less ; I say that the greater circumference BAC is equal to the greater circumference EDF ; and the lesser circumference BGC is equal to the less EHF.

For let K, L the centers of the circles be taken ; and let BK, KC, EL, LF be joined.

And because the circles are equal ; the lines also from their centers are equal (by def. 1.) ; therefore the two BK, KC are equal to the two EL, LF ; and the base BC is (by supp.) equal to the base EF ; wherefore (by 8. 1.) the angle BKC is equal to the angle ELF ; but equal angles stand upon equal circumferences, when they are at the center (by 26. 3.) ; therefore the circumference BGC is equal to the circumference EHF ; but the whole circle ABC is equal to the whole circle DEF ; wherefore the remaining circumference BAC, is equal to the remaining circumference EDF.



Wherefore equal straight lines in equal circles cut off equal circumferences ; the greater *equal* to the greater ; and the less *equal* to the less. Which was to be demonstrated.

P R O P. XXIX.

In equal circles, equal straight lines are extended under equal circumferences.

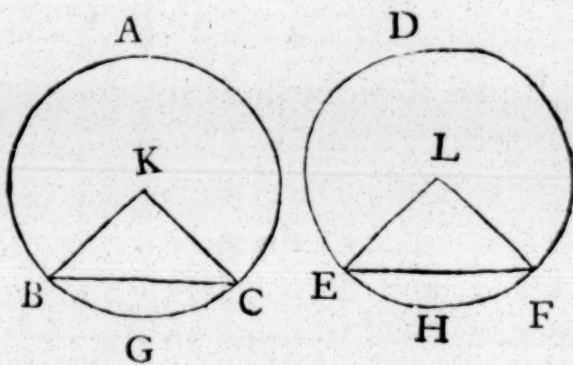
M 2

Let

Book III. Let ABC, DEF be equal circles ; and in them let the equal circumferences BGC, EHF be taken ; and let the straight lines BC, EF be drawn ; I say that the straight line BC is equal to EF .

For let K, L , the centers of the circles be taken ; and let BK, KC, EL, LF be joined.

And because the circumference BGC is equal to the circumference EHF ; also (by 27. 3.) the angle BKC is equal to the angle ELF ; and because the circles ABC, DEF are equal ; the lines from their centers are equal (by def. 1. 3.) ; therefore the two BK, KC are



equal to the two EL, LF ; and they contain equal angles ; therefore (by 4. 1.) the base BC is equal to the base EF .

Wherefore, in equal circles, equal straight lines are extended under equal circumferences. Which was to be demonstrated.

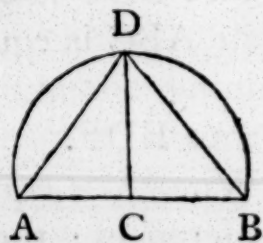
PROP. XXX.

To cut a given circumference in halves.

Let ABD be the given circumference ; it is required to cut the given circumference ADB in halves.

Let AB be joined, and let it be cut in halves (by 10. 1.) at the point C ; and from the point C , let CD be drawn at right angles to the straight line AB ; and let AD, DB be joined.

And since AC is equal to CB ; and CD common ; certainly the two AC, CD are equal to the two BC, CD ; and the angle ACD is equal to the angle BCD ; for each of *these* is a right angle ; and (by 4. 1.) the base AD is equal to the base DB ; but equal straight lines cut off equal circumferences (by 28. 3.) ; the greater *equal* to the greater ; and the less *equal* to the less : and each of the circumferences AD, DB is less than a semicircle ; wherefore the circumference AD is equal to the circumference DB .



Where-

Wherefore the given circumference hath been cut in halves. *Book III.*
Which was to be done.

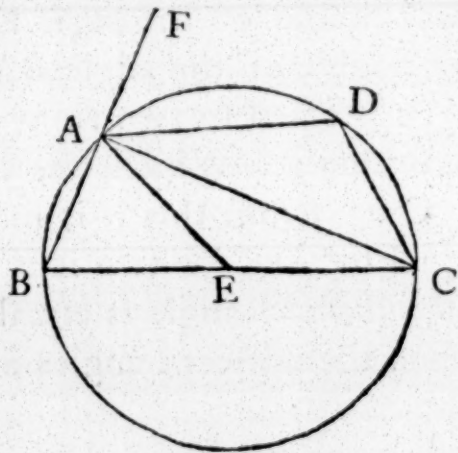
P R O P. XXXI.

In a circle, the angle in a semicircle is a right angle ; but the *angle* in a greater segment is less than a right angle ; and the *angle* in a segment less than a semicircle is greater than a right angle : and besides the angle of a greater segment is greater than a right angle ; but the angle of a segment less than a semicircle is less than a right angle.

Let ABCD be a circle ; and let BC be the diameter of it, and E the center ; also let BA, AC, AD, DC be joined ; I say that the angle in the semicircle BAC is a right angle ; but that the angle in the segment ABC greater than a semicircle, viz. the *angle* ABC, is less than a right angle ; and that the angle in the segment ADC less than a semicircle is greater than a right angle.

Let AE be joined ; and let BA be produced to F.

And because BE is equal to EA, the angle EAB (by 5. 1.) is also equal to EBA : again because EA is equal to EC, the angle EAC is also equal to ACE ; wherefore the whole *angle* BAC is equal to the two *angles* ABC, ACB : but the *angle* FAC, the outward *angle* of the triangle ABC, is also equal to the two angles ABC, ACB (by 32. 1.) : wherefore the angle BAC is equal to the *angle* FAC ; therefore *each of them* is a right angle : wherefore the angle BAC, in the semicircle BAC is a right angle.



And since the two angles ABC, BAC of the triangle ABC are less than two right angles ; and BAC is a right angle : therefore the angle ABC is less than a right angle ; and it is in the segment ABC greater than a semicircle.

And because ABCD is a quadrilateral *figure inscribed* in a circle ; and the opposite angles of quadrilateral *figures* in circles are (by 22. 3.)

Book III. 22. 3.) equal to two right angles ; wherefore the angles ABC, ADC are equal to two right angles ; but ABC is less than a right angle ; therefore the remainder ADC is greater than a right angle ; and it is in a segment less than a semicircle.

I say also that the angle of a greater segment, the angle contained by the circumference ABC and the straight line AC, is greater than a right angle ; but the angle of the segment less than a semicircle, the angle contained by the circumference ADC and the straight line AC is less than a right angle : and it is manifest from hence. For because the angle contained by the straight lines BA, AC is a right angle ; therefore the angle contained by the circumference ABC and the straight line AC is greater than a right angle (by com. not. 9.) : Again because the angle contained by the straight lines CA, AF is a right angle ; therefore the angle contained by the straight line AC and the circumference ADC is less than a right angle.

OTHERWISE. A demonstration that BAC is a right angle. Because the angle AEC (by 32. 1.) is double of the angle BAE ; for it is equal to the two inward and opposite angles ; and AEB is also double of the angle EAC ; wherefore the angles AEB, AEC are double of the angle BAC ; but the angles AEB, AEC are (by 13. 1.) equal to two right angles ; wherefore the angle BAC is a right angle. Which was to be demonstrated.

Cor From this it is manifest that if one angle of a triangle be equal to the other two it is a right angle : for this reason because the adjacent angle is equal to the same two : but (by def. 10. 1.) when the adjacent angles are equal, they are right angles.

P R O P. XXXII.

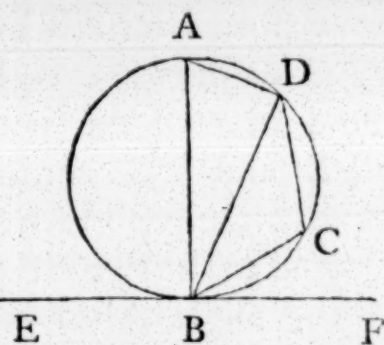
If any straight line touch a circle ; and if from the contact any straight line be drawn to the circle, cutting the circle ; the angles which it makes with the touching line will be equal to the angles in the alternate segments of the circle.

For let any straight line EF touch the circle ABCD in the point B ; and from the point B let BD any straight line be drawn to the circle ABCD cutting it ; I say that the angles which BD makes with

with the touching line EF will be equal to the angles in the alternate segments of the circle, that is, that the angle FBD is equal to the angle constituted in the segment DAB; and the angle EBD is equal to the angle in the segment DCB. Book III.

For let BA be drawn from the point B at right angles to EF; and let C any accidental point be taken in the circumference BD; and let AD, DC, CB be joined.

And because a certain straight line EF touches the circle ABCD in the point B; and from the contact at B the straight line AB has been drawn at right angles to the touching line; the center of the circle ABCD is (by 19. 3.) in the straight line AB: therefore the angle ADB, being in a semicircle, is (by 31. 3.) a right angle; therefore BAD, ABD, the remaining angles, are equal to one right angle: but ABF is also a right angle (by const.); wherefore the angle ABF is equal to the angles BAD, ABD; let the common angle ABD be taken away; therefore the remaining angle DBF is equal to the angle in the alternate segment of the circle, viz. the angle BAD: And because ABCD is a quadrilateral figure inscribed in a circle, its opposite angles are (by 22. 3.) equal to two right angles; therefore the angles DBF, DBE are equal to BAD, BCD; of which BAD has been demonstrated to be equal to DBF; therefore the remaining angle DBE is equal to the angle in the alternate segment DCB of the circle, viz. the angle DCB.



Wherefore if any straight line touch a circle; and if from the contact any straight line be drawn to the circle, cutting the circle; the angles which it makes with the touching line will be equal to the angles in the alternate segments of the circle. Which was to be demonstrated.

P R O P. XXXIII.

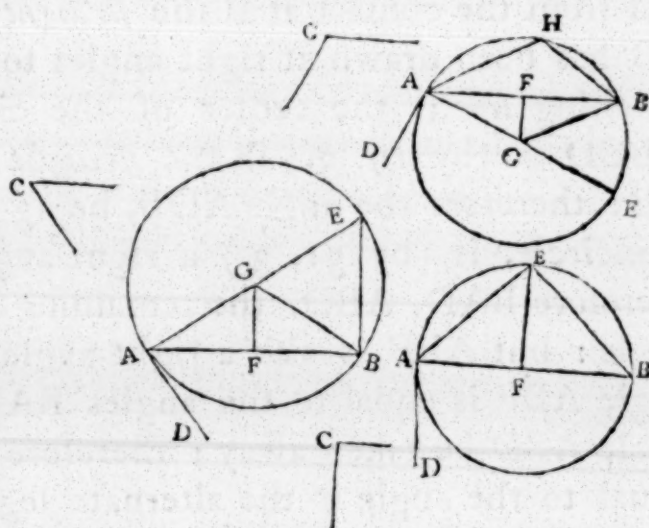
Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let the given straight line be AB; and the angle at C, the given rectilineal angle: it is required, upon the given straight line AB, to

Book III. to describe a segment of a circle containing an angle equal to the angle at C : but the angle at C is either an acute angle or a right angle or an obtuse angle.

First let it be an acute angle, as in the first figure ; and let the angle BAD be made, with the straight line AB and at the point A in it, equal to the angle at C ; wherefore also the angle BAD is acute : and let AE be drawn, from the point A, at right angles to AD ; and let AB be cut in halves at F ; and from the point F let FG be drawn at right angles to AB ; and let GB be joined.

And because AF is equal to FB and FG common ; certainly the two AF, FG are equal to the two BF, FG ; and the angle AFG is (by const.) equal to BFG ; therefore (by 4. 1.) the base AG is equal to the base GB ; therefore the circle described with the center G and at the distance AG, will also pass through B : let it be described ; and let it be ABE ; and let BE be joined. Wherefore because from the extremity of the diameter AE ; from the point A ; AD is at right angles to AE ; therefore AD touches the circle (by cor. to 16. 3.). And since a certain straight line AD touches the circle ABE ; and from the contact at A a certain straight line AB hath been drawn to the circle ABE ; therefore the angle DAB is equal to the angle AEB in the alternate segment of the circle ; but the angle DAB is equal to the angle at C ; therefore the angle at C is equal to the angle AEB : wherefore upon the given straight line AB, a segment of a circle hath been described viz. AEB, containing the angle AEB equal to the given angle at C.



But let the angle at C be a right angle ; and again let it be required to describe upon AB a segment of a circle containing an angle equal to the right angle at C : Again let the angle BAD be made equal to the right angle at C, as it is in the second figure ; and let AB be cut in halves in F ; and with the center F, and at the

the

the distance of either of *the lines* AF, FB let the circle AEB be described. Therefore the straight line AD touches the circle ABE; because the angle at A is a right angle; and the angle BAD is equal to the *angle* in the segment AEB; for it is also a right angle (by 31. 3.) being in a semicircle: but the *angle* BAD is equal to the *angle* at C: wherefore, again a segment of a circle viz. AEB has been described upon AB, containing an angle equal to the *angle* at C. Book III.

But let the *angle* at C be an obtuse *angle*: and let the angle BAD be made equal to it, with the straight line AB and at the point A, as it is in the third figure; and let AE be drawn at right angles to AD; and again let AB be cut in halves at F; and let FG be drawn at right angles to AB; and let GB be joined.

And again, because AF is equal to FB and FG common; certainly the two AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG: therefore the base AG is equal to the base GB; wherefore a circle described with the center G; and at the distance AG, will also pass through B; let it pass as AEB: And because AD has been drawn at right angles to the diameter AE from *its* extremity: therefore AD (by cor. 16. 3.) touches the circle AEB: and from the contact at A, the *straight line* AB hath been drawn; wherefore the angle BAD is equal to AHB the angle contained in the alternate segment of the circle: but the angle BAD is equal to the *angle* at C: therefore also the angle in the segment AHB is equal to the *angle* at C. Wherefore upon the given straight line AB, a segment of a circle; viz. AHB, has been described, containing an angle equal to the *angle* at C. Which was to be done.

P R O P. XXXIV.

To cut off a segment from a given circle, containing an angle equal to a given rectilineal angle.

Let ABC be the given circle; and the given rectilineal angle, the *angle* at D: it is required to cut off a segment from the circle ABC, containing an angle equal to the *angle* at D.

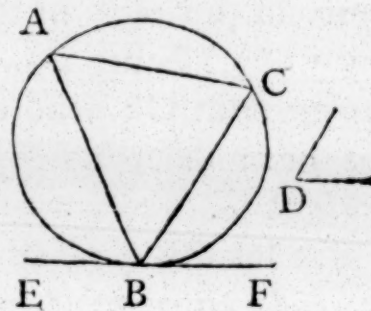
VOL. I.

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Let

Book III. Let EF be drawn touching the circle ABC in the point B; and let the angle FBC be made with the straight line EF, and at the point B in it, equal to the angle at D.

Wherefore because a certain straight line EF touches the circle ABC, and from the contact at B, BC hath been drawn *cutting it*; therefore (by 32. 3.) FBC is equal to the angle contained in the alternate segment BAC: but the *angle* FBC is equal to the angle at D; wherefore also the angle in the segment BAC is equal to the angle at D.



Wherefore from a given circle ABC a segment BAC has been cut off, containing an angle equal to the given rectilineal angle at D. Which was to be done.

P R O P. XXXV.

If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

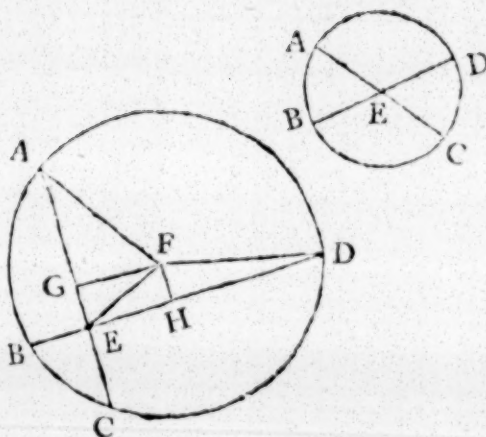
For in the circle ABCD let the two straight lines AC, BD cut one another in the point E; I say that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.

If AC, BD pass through the center so that E be the center of the circle ABCD: it is manifest, AE, EC, DE, EB being equal, that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.

But let AC, DB not pass through the center; and let the center of the circle ABCD be taken (by 1. 3.); and let it be F; and from the point F, let FG, FH be drawn perpendiculars to AC, DB; and let FD, FA, FE be joined.

And because GF a certain straight line through the center cuts a certain straight line AC not through the center at right angles; it will also cut it in halves (by 3. 3.); wherefore AG is equal to GC: wherefore because the straight line AC has been cut into equal segments at the point G; and into unequal segments at the point E; therefore (by 5. 2.) the rectangle contained by AE and EC

EC together with the square of GE is equal to the square of GA; let the square of GF which is common be added: wherefore the rectangle contained by AE, EC together with the squares of GE, GF is equal to the squares of AG, GF: But the square of FE is equal to the squares of EG, GF (by 47. 1.); and the square of FA is equal



to the squares of AG, GF: wherefore the rectangle contained by AE, EC together with the square of FE is equal to the square of FA: but FA is equal to FD; wherefore the rectangle contained by AE, EC together with the square of FE is equal to the square of FD. Certainly for the same reason also the rectangle contained by DE, EB together with the square of FE is equal to the square of FD: but it has also been demonstrated that the rectangle contained by AE, EC together with the square of FE is equal to the square of FD: wherefore the rectangle contained by AE, EC together with the square of FE is equal to the rectangle contained by DE, EB together with the square of FE: let the square of FE which is common be taken away; therefore the remaining rectangle contained by AE, EC is equal to the remaining rectangle contained by DE, EB.

Wherefore if in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other. Which was to be demonstrated.

P R O P. XXXVI.

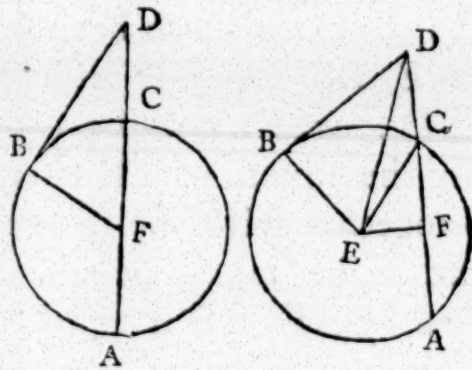
If any point be taken without a circle, and two straight lines fall from it upon the circle; and one of them cuts the circle and the other touches it; the rectangle contained by the whole cutting line and the segment without, taken between the point and the convex circumference, will be equal to the square of the touching line.

For let D, any point without the circle ABC, be taken; and from the point D, let the two straight lines DCA, DB fall upon

Book III. the circle ABC ; and let DCA cut the circle ABC ; and let DB touch it: I say that the rectangle contained by AD , DC is equal to the square of DB : the straight line DCA either passes through the center or not.

First let it pass through the center; and let F be the center of the circle ABC ; and let FB be joined; therefore (by 18. 3.) FBD is a right angle: And because the straight line AC hath been cut in halves in the point F , and CD is added to it; therefore (by 6. 2.) the rectangle contained by AD , DC together with the square of FC is equal to the square of FD : but the square of FD (by 47. 1.) is equal to the squares of FB , BD ; for the angle FBD is a right angle: wherefore the rectangle contained by AD , DC together with the square of FB is equal to the squares of FB , BD ; let the common square of FB be taken away; therefore the remaining rectangle contained by AD , DC is equal to the remaining square of DB the touching line.

But let DA not pass through the center of the circle ABC : and let E the center of the circle be taken; and from E let EF be drawn perpendicular to AC ; and let EB , EC , ED be joined; wherefore EFD is a right angle; and because EF a certain straight line through the center, cuts at right angles AC a certain straight line not through the center, it will also (by 3. 3.) cut it in halves; therefore AF is equal to FC : and because the straight line AC hath been cut in halves at F ; and CD is added to it; therefore (by 6. 2.) the rectangle contained by AD , DC together with the square of FC is equal to the square of FD : let the common square of FE be added; therefore the rectangle contained by AD , DC together with the squares of CF , FE are equal to the squares of DF , FE : but the square of DE is equal to the squares of DF , FE (by 47. 1.); for EFD is a right angle; and the square of CE is equal to the squares of CF , FE : Wherefore the rectangle contained by AD , DC together with the square of EC is equal to the square of ED : but CE is equal to EB ; therefore the rectangle contained by AD , DC together with the square



square of EB is equal to the square of ED: but the square of ED ^{Book III.} is equal to the squares of EB, BD; for the angle EBD is a right angle: wherefore the rectangle contained by AD, DC together with the square of EB is equal to the squares of EB, BD; let the common square of EB be taken away; therefore the remaining *rectangle contained by AD, DC* is equal to the *remaining square of DB*.

. Wherefore if any point be taken without a circle, &c. Which was to be demonstrated.

P R O P. XXXVII.

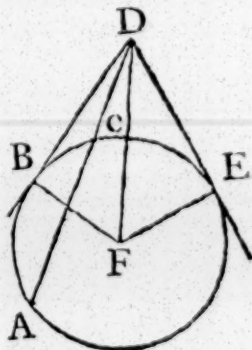
If any point be taken without a circle, and two straight lines fall from the point upon the circle, and one of them cuts the circle and the other meets *it*, and if the *rectangle contained by the whole cutting line, and the segment without*, taken between the point and the convex circumference, be equal to the *square of the line which meets it*; the meeting *line* will touch the circle.

For let D be taken any point without the circle ABC; and let the two straight lines DCA, DB fall from the point D, upon the circle ABC; and let DCA cut the circle, and DB meet *it*; and let the *rectangle contained by AD, DC* be equal to the *square of DB*; I say that DB touches the circle ABC.

For let DE be drawn (by 17. 3.) touching the circle ABC; and let F the center of the circle ABC be taken (by 1. 3.); and let FE, FB, FD be joined: therefore the *angle FED* is (by 18. 3.) a right angle.

And since DE touches the circle ABC; and DCA cuts *it*; therefore (by 36. 3.) the *rectangle contained by AD, DC* is equal to the *square of DE*; but the *rectangle contained by AD, DC* is supposed equal to the *square of DB*; therefore the square of DE is equal to the square of DB; wherefore DE is equal to DB; and FE is also equal to FB; therefore the two DE, EF are equal

to the two DB, BF; and DF is a common base to them; therefore (by 8. 1.) the angle DEF is equal to the angle DBF; but DEF is a right



Book III. a right angle ; therefore DBF is also a right angle ; and FB produced is a diameter ; but the *straight line* drawn from the extremity at right angles to the diameter touches the circle ABC (by cor. to 16. 3.) ; certainly it will be demonstrated in the same manner if the center happen to be in AC.

Wherefore if any point be taken without a circle, &c. Which was to be demonstrated.

THE
ELEMENTS
OF
EUCLID.
BOOK IV.

DEFINITIONS.

1. **A** Rectilineal figure is said to be inscribed in a rectilineal figure, when each of the angles of the inscribed figure touches each side of the figure in which it is inscribed. 2. And in like manner a figure is said to be circumscribed about a figure, when each side of the circumscribed *figure* touches each angle of the figure about which it is circumscribed. 3. And a rectilineal figure is said to be inscribed in a circle when each angle of the inscribed *figure* touches the circumference of the circle. 4. And a rectilineal figure is said to be circumscribed about a circle, when each side of the circumscribed figure (*is a tangent to the circle*) touches the circumference of the circle. 5. In like manner a circle is said to be inscribed in a figure, when the circumference of the circle touches each side of the *figure* within which it is inscribed. 6. But a circle is said to be circumscribed about a figure, when the
- Book IV.
- cir-

Book IV. circumference of the circle touches each angle of the *figure* about which it is circumscribed.

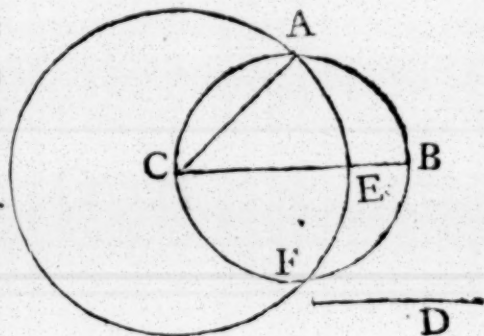
7. A straight line is said to be applied *or placed* in a circle, when the extremities of it are in the circumference of the circle.

P R O P. I.

In a given circle to apply a straight line, equal to a given straight line, which is not greater than the diameter.

Let the given circle be ABC; and D the given straight line, not greater than the diameter of the circle: it is required to apply in the circle ABC a straight line equal to the straight line D.

Let BC the diameter of the circle ABC be drawn: if therefore BC be equal to D, the thing required has been done: for the straight line BC has been applied, in the circle ABC, equal to the straight line D. But if not, BC is greater than D (by supp.); and (by 3. 1.) make CE equal to D; and with the center C, and at the distance CE let the circle AEF be described; and let CA be joined.



Wherefore because the point C is the center of the circle AEF CA is equal to CE; but D is equal to CE; therefore also D is equal to CA.

Wherefore in the given circle ABC, the straight line AC has been applied, equal to the given straight line D, which is not greater than the diameter of the circle. Which was to be done.

P R O P. II.

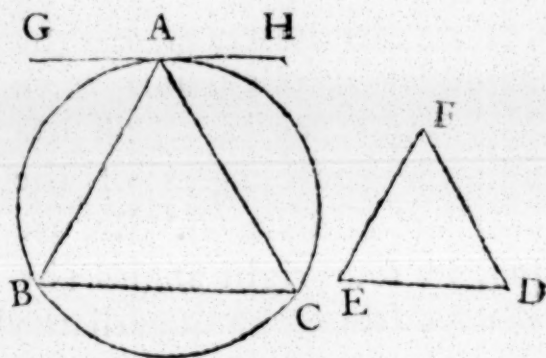
In a given circle, to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.

Let

Let GAH be drawn, touching the circle ABC in the *point* A; ^{Book IV.} and let the angle HAC be made with the straight line AH, and at the *point* A in it, equal to the angle DEF: and the *angle* GAB, with the straight line AG, and at the *point* A in it, equal to the angle FDE; and let BC be joined.

Therefore because a certain straight line HAG touches the circle ABC, and a certain *straight line* AC hath been drawn from the contact *cutting it*; therefore (by 32. 3.) the angle HAC is equal to the *angle* ABC, the angle in the alternate segment of the



circle: but the angle HAC is equal to the angle DEF; therefore the angle ABC is equal to the *angle* FED. Certainly for the same reason also the angle ACB is equal to FDE; and therefore (by 32. 1.) the remaining angle BAC is equal to the remaining angle EFD: wherefore the triangle ABC is equiangular to the triangle DEF; and it has been inscribed in the circle ABC (by def. 3. 4.).

Wherefore a triangle equiangular to the given triangle hath been inscribed in the given circle. Which was to be done.

P R O P. III.

To circumscribe a triangle, about a given circle, equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to circumscribe a triangle about the circle ABC equiangular to the triangle DEF.

Let EF be produced towards both parts, to the points H, G; and let K the center of the circle ABC be taken; and let the straight line KB be drawn as it may happen: and let the angle BKA be made with the straight line KB and at the point K in it equal to the angle DEG; and BKC equal to the angle DFH; and through the points A, B, C let the *straight lines* LAM, MBN, NCL be drawn touching the circle ABC.

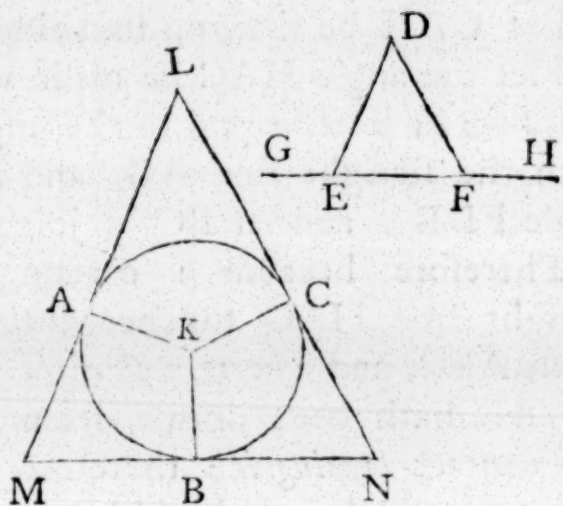
VOL. I.

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Book IV.

And because LM, MN, NL touch the circle ABC in the points, A, B, C; and from the center K; KA, KB, KC have been drawn to the points A, B, C; therefore (by 18. 3.) the angles at the points A, B, C are right angles. And because the four angles of the quadrilateral figure AMBK are equal to four right angles (by 32. 1.); [since the quadrilateral figure AMBK is divisible into two triangles] of which the angles KAM, KBM are two right angles; wherefore the remaining angles AKB, AMB are equal to two right angles; but the *angles* DEG, DEF are also (by 13. 1.) equal to two right angles; wherefore the *angles* AKB, AMB are equal to the *angles* DEG, DEF; of which AKB is (by const.) equal to DEG; therefore the remaining *angle* AMB is equal to the remaining *angle* DEF. Certainly in the same manner it will be demonstrated that the *angle* LNM is equal to the *angle* DFE; therefore the remaining *angle* MLN is equal to the remaining *angle* EDF (by 32. 1.); therefore the triangle LMN is equiangular to the triangle DEF and (by def. 4. 4.) it is circumscribed about the circle ABC.



Wherefore a triangle, has been circumscribed about the given circle, equiangular to the given triangle. Which was to be done.

P R O P. IV.

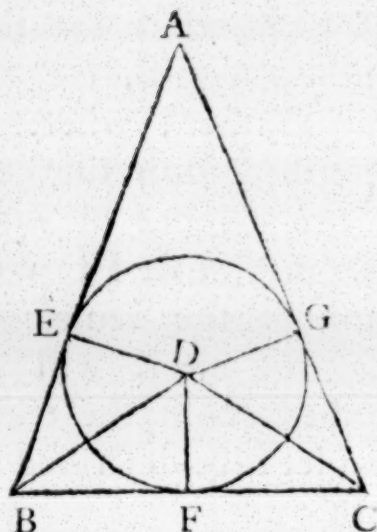
To inscribe a circle in a given triangle.

Let ABC be the given triangle; it is required to inscribe a circle in the triangle ABC.

Let the angles ABC, BCA be cut in halves by the straight lines BD, CD; and let them meet one another in the point D; and from the *point* D let DE, DF, DG be drawn perpendiculars to the straight lines AB, BC, CA.

And because the angle ABD is equal to the angle CBD, for ABC is cut in halves; also the right angle BED is equal to the right

right angle BFD ; therefore there are two triangles EBD, DBF having two angles equal to two angles, and one side equal to one side ; viz. BD common to both extended under one of the equal angles ; also (by 26. 1.) they will therefore have the remaining sides equal to the remaining sides ; therefore DE is equal to DF : Certainly for the same reason also DG is equal to DF ; wherefore the circle described with the center D and at the distance of *any* one of the *lines* DE, DF, DG will pass also through the remaining points, and will touch the straight lines AB, BC, CA ; on account of the angles at the points E, F, G being right angles ; for if it shall cut them ; there will be a *straight line*, drawn at right angles to the diameter from *its* extremity, falling within the circle, which (by 16. 3.) is absurd : wherefore the circle described with the center D and at the distance of *any* one of the *lines* DE, DF, DG does not cut the straight lines AB, BC, CA ; therefore it will touch them ; and will be a circle inscribed in the triangle ABC.



Wherefore the circle EFG is inscribed in the given triangle ABC. Which was to be done.

P R O P. V.

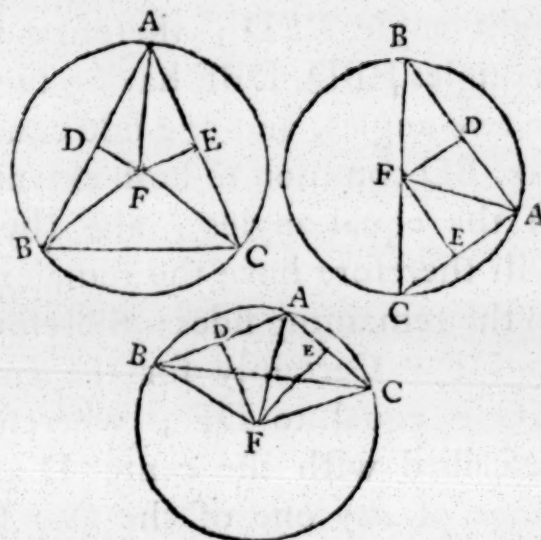
To circumscribe a circle about a given triangle.

Let ABC be the given triangle : it is required to circumscribe a circle about the given triangle ABC.

Let AB, AC be cut in halves in the points D and E ; and let DF and EF be drawn from the points D, E at right angles to AB, AC ; they will meet either within the triangle ABC or in the straight line BC or without the triangle ABC.

First let them meet within at the point F ; and let BF, FC, FA be joined : and because AD is equal to DB ; and DF common and at right angles ; therefore (by 4. 1.) the base AF is equal to the base FB : Certainly in like manner we shall demonstrate that CF is also equal to FA ; so that (by com. not. 1.) BF is also equal to FC ;

Book IV. FC ; therefore the three straight lines FA, FB, FC are equal to one another ; wherefore a circle described with the center F and at the distance of *any* one of the lines FA, FB, FC will also pass through the remaining points ; and the circle will be circumscribed about the triangle ABC ; and let it be described as the circle ABC .



But let DF, EF meet in the straight line BC , as it is in the second figure ; and let AF be joined : Certainly in the same manner we shall demonstrate that the point F is the center of the circle circumscribed about the triangle ABC .

But let DF, EF meet without the triangle ABC , again in the point F ; as it is in the third figure ; and let AF, FB, FC be joined ; and again because AD is equal to DB , and DF common and at right angles ; therefore (by 4. 1.) the base AF is equal to the base FB : Certainly in the same manner we shall demonstrate that CF is equal to FA ; so that also BF is equal to FC ; therefore again the circle described with the center F and at the distance of *any* one of the lines FA, FB, FC will also pass through the remaining points ; and will be circumscribed about the triangle ABC ; and let it be described as ABC .

Wherefore a circle has been circumscribed about the given triangle. Which was to be done.

Cor. And *it is* manifest that, when the center of the circle falls within the triangle, the angle BAC being in a segment greater than a semicircle, is less than a right angle ; but when *it falls* in BC being in a semicircle, it will be a right angle ; but when the center falls without the triangle ABC , the angle BAC being in a segment less than a semicircle, is greater than a right angle. So that also when the given triangle is acute angled the straight lines DF, EF will meet within the triangle ; but when BAC is a right angle ; *they will meet* in BC ; but when it is greater than a right angle, without the triangle ABC .

P R O P.

P R O P. VI.

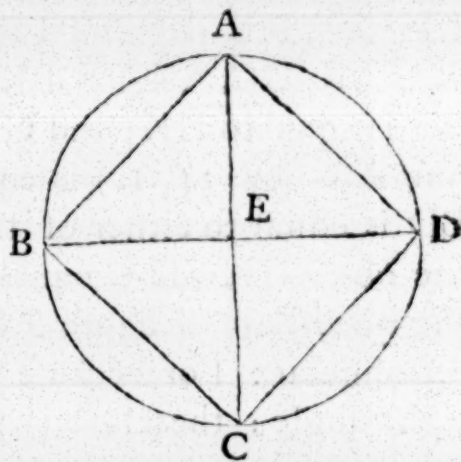
Book IV.

To inscribe a square in a given circle.

Let ABCD be the given circle : it is required to inscribe a square in the circle ABCD.

Let AC, BD, diameters of the circle ABCD, be drawn at right angles to one another ; and let AB, BC, CD, DA be joined.

And because BE is equal to ED ; for E is the center ; and EA common and at right angles ; therefore (by 4. 1.) the base AB is equal to the base AD : certainly for the same reason also each of the *lines* BC, CD is equal to each of the *lines* BA, AD ; therefore the quadrilateral figure ABCD is equilateral.



I say it is also rectangular : for because the straight line BD is a diameter of the circle ABCD ; therefore BAD is a semicircle ; therefore the angle BAD is a right angle (by 31. 3.) : Certainly for the same reason also each of the *angles* ABC, BCD, CDA is a right angle ; therefore the quadrilateral figure ABCD is rectangular : but it has been demonstrated to be equilateral ; therefore it is a square ; and it has been inscribed in the given circle ABCD.

Wherefore the square ABCD has been inscribed in the given circle ABCD. Which was to be done.

P R O P. VII.

To circumscribe a square about a given circle.

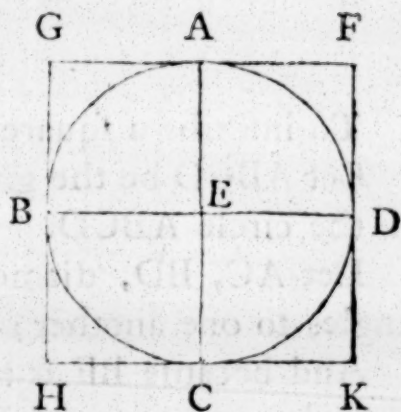
Let ABCD be the given circle : it is required to circumscribe a square about the circle ABCD.

Let AC, BD two diameters of the circle ABCD be drawn at right angles to one another ; and through the points A, B, C, D let FG, GH, HK, KF be drawn touching the circle ABCD.

Wherefore since FG touches the circle ABCD ; and from the center E to the contact at A, the *straight line* AE hath been drawn ; therefore the angles at A are right angles : Certainly for the same reason

Book IV.

reason the angles at the points B, C, D are right angles; and because the angle AEB is a right angle (by const.); as also the angle EBG (by cor. to 16. 3.) is a right angle; therefore (by 29. 1.) GH is parallel to AC: Certainly for the same reason also AC is parallel to FK: and in the same manner we shall demonstrate that each of the lines GF, HK is parallel to BED: wherefore GK, GC, AK, FB, BK are parallelograms; therefore (by 34. 1.) GF is equal to HK; and GH to FK; and because AC is equal to BD; but AC (by 34. 1.) is equal to either of the lines GH, FK; and BD is equal to either of the lines GF, HK; therefore also each of the lines GH, FK is equal to each of the lines GF, HK; therefore the quadrilateral figure FGHK is equilateral: I say that it is also rectangular: For because GBEA is a parallelogram; and AEB is a right angle; therefore (by 34. 1.) AGB is also a right angle; certainly in the same manner we shall demonstrate, that the angles at the points H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular; and it has been also demonstrated to be equilateral; therefore it is a square and has been circumscribed about the circle ABCD.



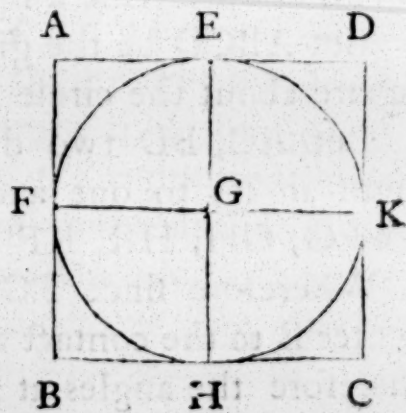
Wherefore a square has been circumscribed about the given circle. Which was to be done.

PROP. VIII.

To inscribe a circle in a given square.

Let ABCD be the given square: it is required to inscribe a circle in the square ABCD.

Let each of the lines AB, AD be cut in halves at the points F, E; and through the point E, let EH be drawn parallel to either of the lines AB, CD; and through the point F, let FK be drawn parallel to either of the lines AD, BC; therefore each of the figures AK, KB; AH, HD; AG, GC; BG, GD is a parallelogram; and certainly (by 34. 1.) the opposite sides of



them are equal : and because AD is equal to AB (by supp.); and AE is the half of AD ; and AF the half of AB ; therefore AE is equal to AF ; so that also the opposite *sides* are equal ; therefore FG is equal to GE : Certainly in the same manner we shall demonstrate that each of the *lines* GH, GK is equal to each of the lines FG, GE ; therefore the four GE, GF, GH, GK are equal to one another ; therefore the circle described with G for a center and at the distance of *any* one of the *lines* GE, GF, GH, GK will also pass through the remaining points ; and will touch the straight lines AB, BC, CD, DA ; on account of the angles at the *points* E, F, H, K being right angles (by cor. to 16. 3.) ; for if the circle will cut the *straight lines* AB, BC, CD, DA ; the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle ; which is absurd (by 16. 3.) ; wherefore the circle described with the center G and at the distance of *any* one of the *lines* GE, GF, GH, GK does not cut the straight lines AB, BC, CD, DA ; therefore it will touch them and will be inscribed in the square ABCD.

Wherefore a circle has been inscribed in a given square. Which was to be done.

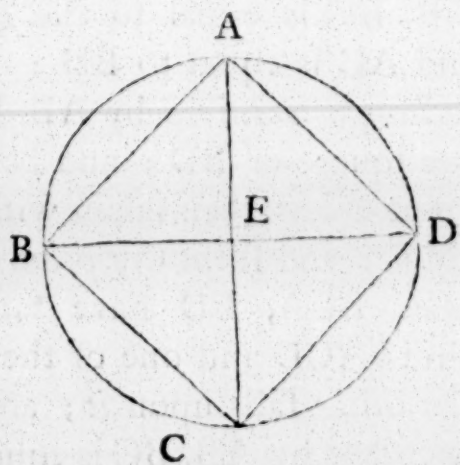
P R O P. IX.

To circumscribe a circle about a given square.

Let ABCD be the given square : it is required to circumscribe a circle about the square ABCD.

For AC and BD being joined let them cut one another in the point E.

And because DA is equal to AB ; and AC common ; certainly the two DA, AC are equal to the two BA, AC and the base DC is equal to the base BC ; therefore (by 8. 1.) the angle DAC is equal to the angle BAC ; therefore the angle DAB has been cut in halves by AC : Certainly in the same manner we shall demonstrate that each of the angles ABC, BCD, CDA hath been cut in



halves

of BD ; wherefore (by 37. 3.) BD touches the circle ACD ; Book IV.
 wherefore because BD touches *it*, and from the contact at D : DC
 has been drawn *cutting it* ; therefore (by 32. 3.) the angle BDC is
 equal to the angle DAC in the alternate segment of the circle :
 wherefore because the *angle* BDC is equal to DAC let the *angle*
 CDA which is common be added ; therefore the whole *angle* BDA
 is equal to the two *angles* CDA, DAC : But (by 32. 1.) the out-
 ward *angle* BCD is equal to the *angles* CDA, DAC ; therefore the
angle BDA is equal to the *angle* BCD : But the *angle* BDA is equal
 to CBD (by 5. 1.) ; since the side AB is equal to AD ; so that also
 the *angle* DBA is equal to the *angle* BCD : therefore the three
angles BDA, DBA, BCD are equal to one another : and since the
angle DBC is equal to BCD (by 6. 1.) the side BD is also equal to
 the side DC ; but BD is put equal to CA ; therefore also AC is
 equal to CD ; so that the angle CDA is equal to DAC ; therefore
 the angles CDA, DAC are the double of DAC ; but the *angle*
 BCD is equal to the *angles* CDA, DAC ; therefore BCD is also the
 double of DAC ; but BCD is equal to either of the *angles* BDA,
 DBA ; therefore each of the *angles* BDA, DBA is double of the
angle DAB.

Therefore an isosceles triangle ADB has been made, having each
 of the angles at the base BD double of the remaining *angle*. Which
 was to be done.

P R O P. XI.

To inscribe an equilateral and equiangular pentagon in a given
 circle.

Let ABCDE be the given circle ; it is required to inscribe an
 equilateral and equiangular pentagon in the circle ABCDE.

Make (by 10. 4.) the isosceles triangle FGH having each of the
 angles at G and H double of the angle at F : and inscribe (by
 2. 4.) the triangle ACD in the circle ABCDE equiangular to the
 triangle FGH ; so that the *angle* CAD may be equal to the angle
 at F ; and each of the *angles* at G and H equal to each of the
 angles ACD, CDA ; therefore also each of the *angles* ACD, CDA
 is double of the *angle* CAD : Let each of the *angles* ACD, CDA
 be cut in halves (by 9. 1.) by the straight lines CE, DB ; and let
 AB, BC, DE, EA be drawn.

Book IV.

Wherefore because each of the angles ACD , CDA is double of the angle CAD ; and they have been cut in halves by the straight lines CE , DB ; therefore the five angles DAC , ACE , ECD , CDB , BDA are equal to one another; but (by 26. 3.) equal angles stand upon equal circumferences; therefore the five circumferences AB , BC , CD , DE , EA are equal to one another: but (by 29. 3.) equal straight lines are extended under equal circumferences; therefore the five straight lines AB , BC , CD , DE , EA are equal to one another; therefore the pentagon $ABCDE$ is equilateral; I say it is also equiangular; for because the circumference AB is equal to the circumference DE ; let the common *circumference* BCD be added; therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$: And the angle AED stands upon the circumference $ABCD$; and the angle BAE stands upon the circumference $EDCB$; and therefore (by 27. 3.) the angle BAE is equal to the angle AED : Certainly for the same reason also each of the angles ABC , BCD , CDE are equal to either of the angles BAE , AED : therefore the pentagon $ABCDE$ is equiangular; but it has also been demonstrated to be equilateral.

Wherefore an equilateral and equiangular pentagon has been inscribed in a given circle. Which was to be done.

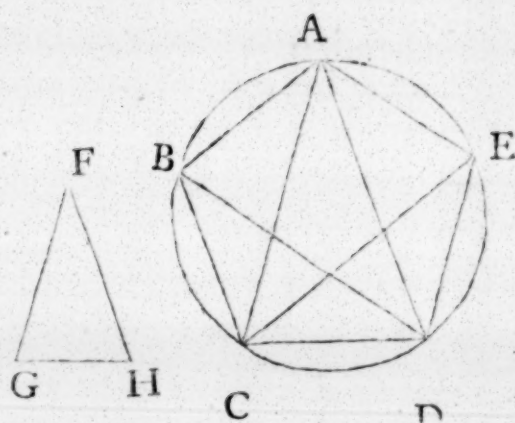
P R O P. XII.

To circumscribe an equilateral and equiangular pentagon about a given circle.

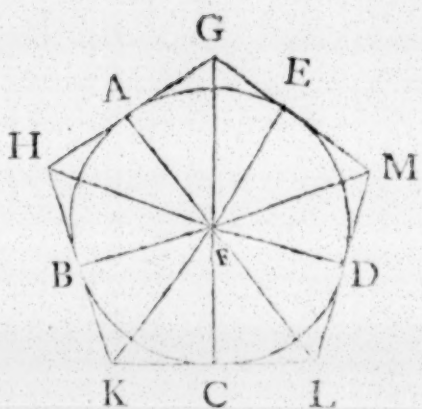
Let $ABCDE$ be the given circle; it is required to circumscribe an equilateral and equiangular pentagon about the circle $ABCDE$.

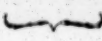
Let A , B , C , D , E be understood to be the points of the angles of a pentagon inscribed (by 11. 4.) so that the circumferences AB , BC , CD , DE , EA may be equal: And through the *points* A , B , C , D , E let GH , HK , KL , LM , MG be drawn touching the circle: and let F the center of the circle $ABCDE$ be taken; and let FB , FK , FC , FL , FD be joined.

And



And because the straight line KL touches the circle $ABCDE$ in the point C ; and from the center F to the contact at C the line FC hath been drawn; therefore (by 18. 3.) FC is perpendicular to KL ; therefore each of the angles at the point C is a right angle. For the same reason also the angles at the points B, D are right angles: and because the angle FCK is a right angle; therefore (by 47. 1.) the square of FK is equal to the squares of FC, CK : Certainly for the same reason also the square of FK is equal to the squares of FB, BK ; therefore (by com. not. 1.) the squares of FC, CK are equal to the squares of FB, BK ; of which the square of FC is equal to the square of FB ; therefore the remaining square of CK is equal to the remaining square of BK ; therefore BK is equal to CK : And because FB is equal to FC and FK common; certainly the two BF, FK are equal to the two CF, FK ; and the base BK is equal to the base CK ; therefore (by 8. 1.) the angle BFK is equal to the angle CFK ; and the angle BKF ; equal to FKC ; therefore the angle BFC is double of KFC and BKC is double of FKC . Certainly for the same reason also CFD is double of CFL ; and the angle CLD is double of CLF . And because the circumference BC is equal to the circumference CD (by the const.); the angle BFC is also equal (by 27. 3.) to the angle CFD ; and BFC is the double of KFC ; and the angle DFC is the double of LFC ; therefore the angle KFC is equal to CFL ; certainly FKC, FLC are two triangles having the two angles equal to the two angles, each to each; and one side equal to one side; FC common to them both; therefore (by 26. 1.) they will also have the remaining sides equal to the remaining sides; and the remaining angle equal to the remaining angle; therefore the straight line KC is equal to CL ; and the angle FKC to FLC : and since KC is equal to CL ; therefore KL is the double of KC . Certainly in the same manner HK will be demonstrated to be double of BK : and because BK has been demonstrated to be equal to KC ; and KL is the double of KC ; and HK the double of BK ; therefore HK is equal to KL ; certainly in the same manner each of the lines



Book IV.  GH, GM, ML will be demonstrated to be equal to either of the lines HK, KL; therefore the pentagon GHKLM is equilateral. I say that it is also equiangular; for because the angle FKC is equal to FLC; and HKL has been demonstrated to be double of FKC; and KLM to be double of FLC; therefore HKL is also equal to KLM: certainly in the same manner each of the angles KHG, HGM, GML will be demonstrated to be equal to either of the angles HKL, KLM; therefore the five angles GHK, HKL, KLM, LMG, MGH are equal to one another: therefore the pentagon GHKLM is equiangular; but it has been also demonstrated to be equilateral; and it has been circumscribed about the given circle ABCDE. Which was to be done.

P R O P. XIII.

To inscribe a circle, in a given pentagon which is equilateral and equiangular.

Let ABCDE be the given pentagon, which is equilateral and equiangular: it is required to inscribe a circle in the pentagon ABCDE.

Let each of the angles BCD, CDE be cut in halves, by each of the straight lines CF, DF; and from the point F, in which the straight lines CF, DF meet one another, let the straight lines FB, FA, FE be drawn. And because BC, CD are equal (by supp.); and CF common; certainly the two BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF (by const.); therefore (by 4. 1.) the base BF is equal to the base DF; and the triangle BFC is equal to the triangle DCF; and the remaining angles are equal to the remaining angles, under which the equal sides are extended; therefore the angle CBF is equal to the angle CDF: and because the angle CDE is double of CDF; but CDE is equal (by supp.) to ABC; and the angle CDF to CBF; therefore CBA is the double of CBF; therefore the angle ABF is equal to FBC: therefore the angle ABC is cut in halves by the straight line BF. Certainly in the same manner it will be demonstrated, that each of the angles BAE, AED hath been cut in halves by each of the straight lines FA, FE.

Let

Let the perpendiculars FG , FH , FK , FL , FM be drawn from the point F to the straight lines AB , BC , CD , DE , EA ; and because the angle HCF is equal to KCF ; and the right angle FHC is equal to the right angle FKC ; certainly there are two triangles FHC , FKC having two angles equal to two angles; and one side equal to one side, viz. FC common to them *both*, extended under one of the equal angles; therefore (by 26. 1.) they will have the remaining sides equal to the remaining sides; therefore the perpendicular FH is equal to the perpendicular FK : certainly in the same manner it will be demonstrated, that each of the *lines* FL , FM , FG is equal to either of the *lines* FH , FK ; therefore the five straight lines FG , FH , FK , FL , FM are equal to one another: wherefore the circle described with the center F and at the distance of *any* one of the *lines* FG , FH , FK , FL , FM will also pass through the remaining points and will touch the straight lines AB , BC , CD , DE , EA ; on account of the angles at the points G , H , K , L , M being right angles; for if it will not touch them, but will cut them; it will happen that a *straight line* drawn at right angles to a diameter from its extremity falls within the circle, which (by 16. 3.) has been demonstrated to be absurd: therefore the circle described with the center F and at the distance of *any* one of the *lines* FG , FH , FK , FL , FM will not cut the straight lines AB , BC , CD , DE , EA ; therefore it will touch them: let it be described as $GHLKM$.

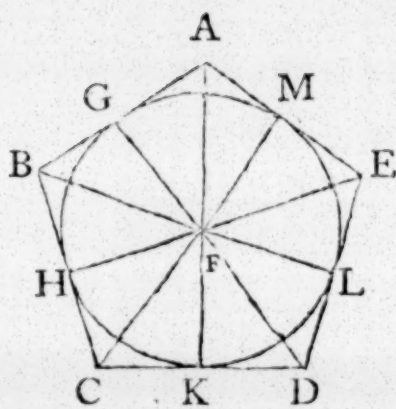
Wherefore a circle has been inscribed in a given pentagon, which is equilateral and equiangular. Which was to be done.

P R O P. XIV.

To circumscribe a circle about a given pentagon, which is equilateral and equiangular.

Let $ABCDE$ be the given pentagon, which is equilateral and equiangular; it is required to circumscribe a circle about the pentagon $ABCDE$.

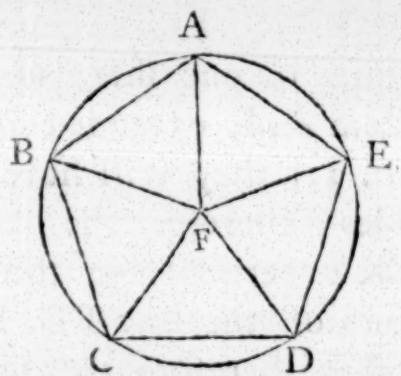
Let



Book IV.

Let each of the angles BCD, CDE be cut in halves by each of the straight lines CF, DF; and from the point F in which the straight lines meet, let the straight lines FB, FA, FE be drawn to the points B, A, E; certainly in the same manner as in the *proposition* before this it will be demonstrated, that each of the angles CBA, BAE, AED hath been cut in halves by each of the straight lines BF, FA, FE.

And because the angle BCD is equal to the angle CDE; and the *angle* FCD is the half of BCD; and the *angle* CDF is the half of CDE; therefore also the *angle* FCD is equal to FDC: so that (by 6. 1.) the side FC is equal to the side FD. Certainly in the same manner it will be demonstrated that each of the *lines* FB, FA FE is equal to either of the *lines* FC, FD: therefore the five straight lines FA, FB, FC, FD, FE are equal to one another; wherefore the circle described with the center F and at a distance of *any* one of the lines FA, FB, FC, FD, FE will also pass through the remaining points; and will be circumscribed about the pentagon ABCDE, which is equilateral and equiangular; let it be circumscribed and let it be ABCDE.



Wherefore a circle has been circumscribed about a given pentagon, which is equilateral and equiangular. Which was to be done.

PROP. XV.

To inscribe an equilateral and equiangular hexagon in a given circle.

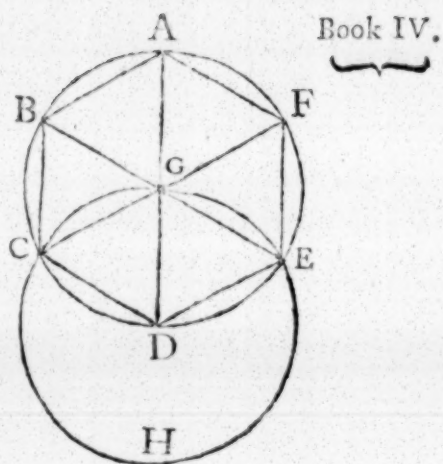
Let ABCDEF be the given circle; it is required to inscribe an equilateral and equiangular hexagon in the circle ABCDEF.

Let AD the diameter of the circle ABCDEF be drawn: and let G the center of the circle be taken; and with the center D and at the distance DG let the circle EGCH be described; and EG, CG being joined, let them be produced to the points B, F; and let AB, BC, CD, DE, EF, FA be drawn; I say that the hexagon ABCDEF is equilateral and equiangular.

For

For because the point G is the center of the circle $ABCDEF$, the *straight line* GE is equal to GD : again because the point D is the center of the circle $EGCH$, the *line* DE is equal to GD ; but GE has been demonstrated to be equal to GD ; therefore GE is equal to ED ; therefore the triangle EGD is equilateral; and therefore the three angles of it EGD , GDE , DEG are equal to one another, since the angles at the base of isosceles triangles are equal to one another; and (by 32. 1.) the three angles of a triangle are equal to two right angles; therefore the angle EGD is the third *part* of two right angles: Certainly in the same manner it will be demonstrated that the angle DGC is the third *part* of two right angles; and because the straight line CG standing upon EB makes the adjacent angles EGC , CGB equal to two right angles; therefore the remaining *angle* CGB is the third *part* of two right angles; wherefore the angles EGD , DGC , CGB are equal to one another; so that (by 15. 1.) the vertical angles to them; viz. BGA , AGF , FGE are equal to EGD , DGC , CGB ; therefore the six angles EGD , DGC , CGB , BGA , AGF , FGE are equal to one another; but (by 26. 3.) equal angles stand upon equal circumferences; therefore the six circumferences AB , BC , CD , DE , EF , FA are equal to one another; but (by 29. 3.) equal straight lines are extended under equal circumferences; therefore the six straight lines are equal to one another; therefore the hexagon $ABCDEF$ is equilateral: I say it is also equiangular; for because the circumference AF is equal to the circumference ED ; let the common circumference $ABCD$ be added; therefore the whole circumference $FABCD$ is equal to the whole circumference $EDCBA$; and the angle FED stands upon the circumference $FABCD$; and the angle AFE upon $EDCBA$; therefore (by 27. 3) the angle AFE is equal to the *angle* DEF ; in like manner it will be demonstrated that the remaining angles of the hexagon $ABCDEF$, one by one, are equal to either of the angles AFE , FED ; therefore the hexagon $ABCDEF$ is equiangular; and it has been demonstrated to be equilateral; and it has been inscribed in the circle $ABCDEF$.

Wherefore an equilateral and equiangular hexagon has been inscribed in a given circle. Which was to be done.



Book IV. Cor. From this *it is* manifest, that the side of a hexagon is equal to the *line* from the center of the circle.

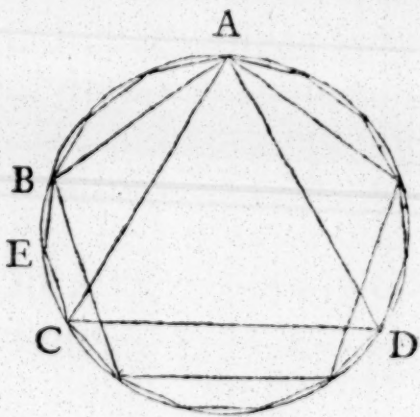
And if we draw through the points A, B, C, D, E, F *straight lines* touching the circle, an equilateral and equiangular hexagon will be circumscribed about the circle, agreeable to what has been said concerning the pentagon: And farther we shall inscribe in a given hexagon and also circumscribe a circle by the like *steps* as have been mentioned concerning the pentagon.

P R O P. XVI.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inscribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC, be inscribed in the circle ABCD, the side of an equilateral triangle inscribed in it; and let AB, be *inscribed*, the side of an equilateral pentagon. Wherefore of what equal segments the circle ABCD is fifteen; of such the circumference ABC, being the third *part* of the circle, will be five; and the circumference AB, being the fifth *part* of the circle, will be three; therefore the remainder BC will be two of these equal *segments*; let BC be cut in halves in the point E (by 30. 3.); therefore either of the circumferences BE, EC is the fifteenth *part* of the circle ABCD; therefore joining the straight lines BE, EC; if we apply in the circle ABCD continually straight lines equal to them (by 1. 4.) there will be inscribed in it an equilateral and equiangular quindecagon. Which was to be done.



And in like manner as was done concerning the pentagon; if through the divisions of the circle we draw *lines* touching the circle; an equilateral and equiangular quindecagon will be circumscribed about the circle; and farther we shall inscribe in a given quindecagon, which is equilateral and equiangular, and also circumscribe a circle by the like *steps* as have been mentioned concerning the pentagon.

DISSERTATION VI.

IN the former dissertation many things are not mentioned which a careless reader requires to be put in mind of, because I considered that such readers would stand in need of fresh admonitions, to induce them to read my observations; which determined me to steer a kind of middle course, by reducing my remarks to some general heads, and illustrating them by examples, which the reader is supposed to apply in all similar circumstances. Upon this principle it seemed needless to remark that the demonstrations of the twelfth, thirteenth and fourteenth propositions of the fourth book, are more general than they appear to be at first sight; for if the reader observe my general rule for the examination of every supposition; he will find no consequence drawn from the number of sides being five; it being only necessary that the inscribed figures be equilateral and equiangular; which shews the demonstrations to be much more general than is professed in the propositions. In like manner, as it has been particularly remarked before, it seemed unnecessary to desire the student to observe the ingenious contrivance for cutting the angles of the figure in halves, in the thirteenth proposition, having earnestly recommended an attention to all such indirect constructions: for, in this instance if all the angles were cut in halves directly, it would be found no easy matter to prove that the lines will all meet in F; but by cutting any two of the angles in halves, which follow each other in order, the point F is fixed; and then joining this point, and the other angular points, it is easy to prove that these lines will cut all the other angles in halves. And whoever reads the former dissertations with this allowance, will readily grant that I have been sufficiently particular. I now proceed to a subject which will exercise

cise the reader's patience if he chuses to go along with me ; for this dissertation will consist of a minute enquiry into the origin of our ideas of proportional magnitudes.

CHAP. I.

Of parts and multiples.

IN the first four books Euclid considers no other relation of magnitudes but their equality ; at least when he speaks of any other, it is in a loose and undetermined manner, without ever considering how much greater or how much less the one is than the other. And this will be obvious to any one who chuses to turn his thoughts to the nature of his propositions ; the common notion of whole and part therefore is sufficiently accurate for his purpose in these books. But the object which he has in view in this book, being to settle other relations of magnitudes besides their equality, makes it necessary to introduce, and consequently to define, a new idea of whole and part under the name of a multiple and part. Take two straight lines, one of them fixt and the other undetermined ; by the third proposition of the first book two unequal lines being given, you may cut off a part from the greater equal to the less ; and thus you may make a straight line, twice, three times, four-times, or fivetimes &c. as long as another ; the line originally fixt is called a part ; and the line which you determine by this construction is called a multiple ; they are relative terms, or as Euclid expresses it, a magnitude of a magnitude. The first thing therefore which the student has to settle in his own mind, is, what the magnitudes are and in what circumstances they must be, before this relation of multiple and part can take place ; and not only so, but what the magnitudes are between which he can exhibit these relations scientifically from what has been demonstrated in the first four books. And first it is obvious that a straight line cannot be a part or multiple of a triangle ; nor of the circumference of a circle ; nor indeed can we consider one triangle as a part of another unless they be between the same parallel lines ; nor the circumferences

rences of two circles; as part and multiple unless the circles be equal: but to settle this more particularly let us consider what magnitudes, we can multiply, that is measure exactly, and in what circumstances: Now we can multiply or measure exactly, two straight lines by the third proposition of the first book; that is we can take any straight line five, six &c. times; or if the supposition be that one straight line is five, six &c. times any other line; we can divide this multiple into its parts. But the reader is not to take my word for this; and the more instances in which he tries it, the more likely is he to understand what follows.

By multiplying (that is doubling, trebling &c.) the base of a triangle, you multiply the triangle, provided the part and multiple are between the same parallels; and the same may be said of parallelograms: so that here we begin to make our notions of magnitudes somewhat more complicated. It is said in the first book that triangles upon equal bases and between the same parallels are equal, but now we advance a step farther, and say that triangles upon double bases are double, and upon treble bases are treble &c; which transition the student who is desirous to have his ideas keep pace with his apparent progress in the science will do well to observe.

Angles, circumferences and sectors of equal circles may be multiplied by the principles contained in the twenty-sixth-seventh-eighth, and-ninth propositions of the third book: because by placing or applying equal straight lines in a circle you cut off equal circumferences; you make equal angles at the center or circumference; and it is also easy to prove that the sectors are equal: all this however the student must demonstrate; and I believe it will be found that these are all the magnitudes which can be multiplied according to Euclid's idea in his two first definitions; for it does not appear to me that it would be sufficient to have one triangle three times as large as another unless they be between the same parallel lines, to say that the one is a multiple of the other; because the definitions suppose that the multiple may be divided into its parts, which is what I understand by the word *measures*. But, all these, the student should examine by a particular construction, if he means to understand any part of this business; and I

am persuaded he will find, that straight lines ; triangles and parallelograms between the same parallel lines ; angles, circumferences and sectors of equal circles are all the magnitudes which can have the relation of multiple and part to one, who has no other principles to go upon but those contained in the first four books.

It may be thought that particular instances explaining each of these circumstances would have been useful ; but the construction of such figures is extremely obvious, and it will be much more improving for the learner to describe the figures for himself, because he may look at figures already described, without comprehending what they are intended to illustrate, which can never be the case if the figures are properly described by himself. Now this being perfectly understood, and the proper meaning of the two first definitions determined ; the student is next to endeavour to acquire a ready use of them ; by reasoning upon such consequences as follow most directly from them : let him suppose one magnitude to be a part of another and try what consequences will follow ; and the only direct consequence is that the part will measure the multiple ; but what is the meaning of this ? Let the reader suppose two straight lines which we shall call AB and C ; and let C be a part of AB ; then if we begin at the point A and (by 3. 1.) take of a part equal to C and from the remainder a part equal to C ; we must at last fall in with the point B ; because, if this did not happen, C could not be a part of AB, contrary to the supposition ; because this is what I understand by measuring it : and the same consequence follows if we suppose AB a multiple of C.

But it will be necessary to reason in the same manner upon each of the magnitudes, which have been mentioned as coming under these definitions of a part and multiple.

CHAP. II.

Of equimultiples.

THE word *equimultiple*, one might imagine has its meaning fully ascertained, by the common acception of the two words of which it is made up ; and yet I cannot tell how it happens, that
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it hardly ever conveys a distinct idea to the mind of a learner, as it is applied in this science; surely nothing seems more obvious than that it is to be understood of magnitudes taken an equal number of times; but I have generally found that there is some confused notion of equality of magnitude that accompanies the conception of equimultiples; than which nothing can possibly be more absurd; for it means only that the magnitudes are taken the same number of times; thus if any straight line, which you may call A be taken three times; and call this BC: and suppose any angle to be trebled; we say that BC and this angle, are equimultiples of the line A and of the angle which was taken three times; or to be more particular (turn to the figure for the first proposition of the sixth book) the straight line HC and the triangle AHC, though they can have no connexion with each other as to magnitude, are nevertheless equimultiples of the straight line BC and of the triangle ABC: and again (see the figure to the thirty third proposition of the sixth book) the circumference EN and the angle EHN, though magnitudes of quite different kinds, are equimultiples of the circumference EF and of the angle EHF; or in short this expression means no more, but that when the multiples are divided into their parts, the number of divisions in each is the same; as in the instances above, HC is divided into three parts each equal to BC; and the triangle AHC into three parts also each equal to the triangle ABC; these equimultiples therefore agree in nothing but that they can be divided into the same number of parts; and this is the most direct consequence which will follow from the supposition of their being equimultiples. But in order to understand the meaning of this term, and the two first definitions it will be proper for the reader to examine the three first propositions together with the fifth and sixth of this book.

C H A P. III.

Containing further illustrations of the first and second definitions.

THE reader will perceive, if he has gone through the different steps of the constructions which I have recommended, that all
those

those magnitudes which we can multiply; are doubled, trebled &c. by the assistance of a straight line; so that whatever can be shewn to be a property of, a multiple, equimultiples or parts of a straight line the same may be readily transferred to triangles or parallelograms, between the same parallel lines; to angles; circumferences and sectors of equal circles; which I contend are all the magnitudes which we can multiply, agreeable to Euclid's two first definitions; and probably for this reason the constructions and demonstrations in this fifth book are confined to straight lines only; although the enunciations are expressed of magnitudes in general. Let us suppose that the reader has examined the three first propositions and the fifth and sixth; only observing, that Euclid is here demonstrating properties of straight lines, I shall now proceed to make some remarks upon them in order. And first it is said that AB is a multiple of E; *equimultiple you mean*; no, not so fast let us examine one thing at a time; I say that the supposition is; AB is a multiple of E; well let us see what consequence follows from this; cut off from AB the line AG equal to E; and by proceeding thus you must fall in with the point B; that is I thus divide AB into parts equal to E; or I measure AB; take care to distinguish between this, *and the supposing it divided*; for if the reader be given to suppositions of this kind, he had better suppose the whole book demonstrated. In like manner CD by the supposition is a multiple of F; and the same consequences follow; namely that in the same manner I divide CD into parts equal to F: But farther by the supposition AB and CD are equimultiples of E and F; let us see what consequence follows from this part of the supposition; nothing more than this, that after I have divided the multiples into magnitudes equal to the parts, the number of divisions in both is the same; for certainly if AB be the double of E and CD the double of F; AB contains E twice and CD contains F twice; and the same may be said of any other equimultiples; and the use made of it in this proposition is, that if you take away the parts one by one, when you have done with one multiple all the other equimultiples must be exhausted at the same time.

But farther, this proposition should be examined still more particularly; because, although this may seem needless for those to whom

whom Euclid writes, I know it to be essentially necessary for those to whom this dissertation is addressed. The student is therefore to bring this conclusion more directly to the definitions, at least to take such steps as seem most likely to carry his attention up to the definitions. For this purpose let him draw two indefinite straight lines; and cutting off from one of them a line equal to E and F together; he is then to proceed in the same manner and cut off from the other a line equal to AG and CH together; and from the remainder a line equal to GB and HD together; and so on until he has made a line equal to AB and CD together: He is next to demonstrate that the line, made equal to E and F together, will measure the line made equal to AB and CD together; and when this is done he can say, that the one line is a part or multiple of the other: But after all it remains to be proved that the one line is contained in the other, just as often as E is contained in AB; and then he may say that AB; and AB, CD together are equimultiples of E; and of E, F together. Let him, after this point is settled to his full conviction, proceed in the same manner with three multiples &c. until he can say upon good grounds if there be any number of magnitudes &c.

The second and third propositions must be perfectly intelligible to any one who has gone the progress which I have recommended upon the first. But before entering upon the fifth it will be proper for the reader to observe that we are not as yet taught how to take even a third part of a straight line, much less how to take any part: indeed we can cut a straight line in halves; by which construction we may take a half, a fourth part, and an eighth part &c. but nothing more: the construction therefore in the fifth proposition is impracticable; for if AB be three times CD; I cannot take a third part of BE but upon the supposition that EB is a multiple of FD; which is taking for granted the very thing to be proved: if a line be a multiple of another I can divide it into its parts; but not otherwise; which makes a different construction necessary here, from that given in the text; but sufficiently obvious from the construction in the next proposition: for producing BA directly forward, we may cut off lines equal to FD, until we have the same multiple of FD, that AB is of CD; and call this AG;
then

then by the first proposition EG is the same multiple of CD that AB is of CD, therefore (by com. not 6.) AB and EG are equal; take away AE which is common, and EB is equal to AG; therefore EB is the same multiple of FD that AB is of CD.

It is not easy to conjecture how this blunder could have crept in, unless by the deception of a particular figure: for if AB be the double of CD, the construction is practicable, because then I divide EB in halves; and take CG equal to the half of it; and then indeed AB is the same multiple of GF as of CD, therefore GF and CD are equal &c. And it is obvious that if AB be taken four times CD the same construction will hold; but it fails, if AB be taken three times, for then I have no principle upon which I could pretend to divide EB; unless I should measure it by FD, which is taking the very thing for granted which we want to demonstrate, namely that EB is a multiple of FD; and not only a multiple of it, *but the same multiple of it*, that AB is of CD; that is, I set out with a supposition that CG is equal to FD in order to prove them to be equal; which is certainly very bad reasoning if it deserve the name. I have been the more particular in explaining this, because it seems well calculated to illustrate this doctrine of parts and multiples.

C H A P. IV.

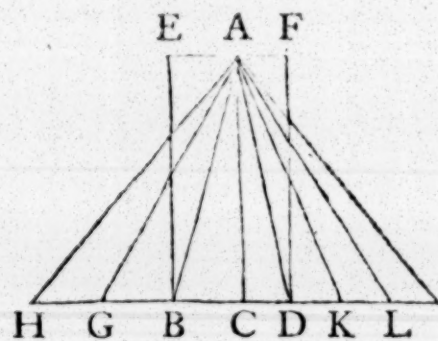
Containing a comparison of the multiples of four magnitudes.

LET ABC, ACD be two triangles between the same parallel lines; let equimultiples of the base BC and of the triangle ABC be taken; and also any equimultiples whatever of the base CD and of the triangle ACD; I say that if the multiple of BC exceed the multiple of CD; the multiple of the triangle ABC will exceed the multiple of the triangle ACD; if equal, equal; and if less, less.

Here the reader is to fix his attention upon four magnitudes, which he may distinguish thus, by calling BC the first, CD the second, the triangle ABC the third, and the triangle ACD the fourth; according to our supposition then, we are to take equimultiples

multiples of the first and third at a venture; and also any equimultiples whatever of the second and fourth. But observe we do not compare the *equimultiples* with one another; only the multiple of the first, with the multiple of the second, which are both straight lines; and the multiples of the third and fourth, which are both triangles. And first let us exhibit any multiple of BC the first at a venture; produce CB indefinitely and cut off as many lines as you please each equal to CB; as BG, GH &c; and let CH be a multiple of BC taken at a venture, which it is if we draw no consequence from CH's containing CB any particular number of times. Now we are to take the triangle ABC the same number of times; and this is performed by joining AH, for the triangle AHC contains the triangle ABC

just as often as CH contains CB; which may be proved by joining AG; for as many lines CB, BG, GH as there are, just so many triangles are there; and because these straight lines are equal; (by 38. 1.) the triangles are equal of which these lines are the bases; and the number of triangles, and the number of



straight lines being the same; CH contains CB just as often as the triangle AHC contains the triangle ABC; therefore CH and the triangle AHC are equimultiples of CB and of the triangle ABC. Again we are to take equimultiples any whatever of the second and fourth, of CD and of the triangle ACD: produce CD directly forward, and take DK, KL each equal to CD; and in the same manner CL is any multiple of CD; and the triangle ACL the same multiple of the triangle ACD. The construction being thus finished let us proceed to the demonstration.

Now I say that whenever CH is less than CL; the triangle AHC is less than the triangle ACL; because I may cut off from CL a part equal to CH; and then joining the extremity of that line to the point A; there will be a triangle, equal to ACH within the triangle ACL; therefore when CH is less than CL, the triangle AHC will always be less than the triangle ACL; and if CH be equal to CL, then (by 38. 1.) the triangle AHC is equal

to ACL : and lastly if CH be greater than CL , it may in like manner be proved that the triangle AHC is greater than the triangle ACL : and all this will be true according to any multiplication whatsoever. And if for the sake of distinctness we call BC the first; CD the second; the triangle ABC the third; and the triangle ACD the fourth; we have this general conclusion; if two such triangles be between the same parallel lines; and if you take any equimultiples whatever of the first and third; and also any equimultiples whatever of the second and fourth; whenever the multiple of the first exceeds the multiple of the second the multiple of the third will always exceed the multiple of the fourth: and when equal, equal: and when less, less. Which was to be demonstrated.

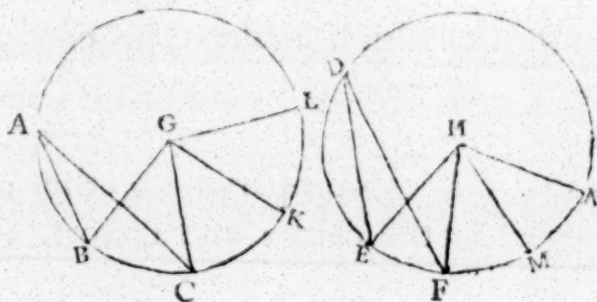
The reader is now to proceed by himself and prove the same thing of the equimultiples of BC and of the parallelogram CE ; and the equimultiples of CD and of the parallelogram CF ; that is, that whenever the multiple of BC exceeds the multiple of CD ; then the multiple of the parallelogram EC will exceed the multiple of the parallelogram CF : and when equal, equal: and when less, less.

Here our reasoning seems to become somewhat more complicated than in the first four books: but if we attend carefully to the meaning of the terms, we shall find very little difference: for the multiple is composed by a repetition of the *same* magnitude; which may therefore be considered only as a comparison of two distinct things: the term *equimultiples* implies only a division of the magnitudes into the same number of equal parts: and even in this last comparison, we have only two magnitudes to compare with one another, as to their being greater; equal; or less. So that if the student but chuses to prepare himself properly, he stands upon the same firm ground here, as in the first four books: for the magnitudes are the same, only the mode of comparison is new: Which must be made familiar by particular instances. Therefore let us suppose two equal circles ABC , DEF ; and BGC , EHF angles at their centers, standing upon the circumferences BC , EF : I say, if you take equimultiples of the circumference BC and of the angle BGC at a venture; and also any equimultiples of the circumference

cumference EF and of the angle EHF : if the multiple of BC exceed the multiple of EF ; I say the multiple of the angle BGC will always exceed the multiple of the angle EHF ; and if equal, equal : and if less, less.

For take any multiple of the circumference BC, by placing or applying in the circle straight lines, equal to a straight line joining B and C ; and thus making the circumferences CK, KL equal to the circumference BC (by 28. 3.) : and thus BL may represent any multiple of the circumference BC : but because the circumferences BC, CK, KL are equal to one another, (by 27. 3.) the angles BGC, CGK, KGL are

equal to one another ; therefore the angle BGL is a multiple of the angle BGC ; but moreover it is the same multiple of this angle, that the circumference BL is of BC ; therefore the circumference BL and



the angle BGL are equimultiples of the circumference BC and of the angle BGC : And in the same manner we may shew that the circumference EN and the angle EHN are equimultiples of the circumference EF and of the angle EHF : now supposing as in the last instance, the circumference BC to be the first ; the circumference EF the second ; the angle BGC the third ; the angle EHF the fourth ; I say that whenever the circumference BL, the multiple of the first, exceeds the circumference EN, the multiple of the second ; then the angle BGL the multiple of the third, will exceed the angle EHN the multiple of the fourth ; for, because BL exceeds EN, I may take of from BL a part equal to EN ; by placing in the circle a straight line equal to the straight line joining the points E and N ; and can thus prove that a part of the angle BGL is equal to EHN : and also if BL be equal to EN the angle BGL is equal to the angle EHN ; and in the same manner it may be proved that if the multiple BL be less than EN, then the angle BGL is less than the angle EHN ; and that all this will happen according to any multiplication whatever &c.

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But here a difficulty seems to occur, because the multiple may be so taken that BG and GL will become one straight line. Also in taking a multiple of a straight line ; there is obviously no bounds set to the operation ; which there seems to be in taking a multiple of a circumference ; though this may be rendered indefinite by describing as many equal circles as we please. Now what I take to be Euclid's meaning, as being most consistent with the common notion of an angle, is this, that in comparing the multiples of the circumferences, all the semicircumferences are to be neglected ; and in comparing the multiples of the angles at the center ; we are only to consider, that part of the angle which exceeds two right angles ; which makes the construction and figure here given general. I would recommend it to the student, to demonstrate the same thing, of the multiples of the circumferences and sectors of equal circles compared together ; as also, of the multiples of the angles at the center and sectors.

C H A P. V.

The same subject continued.

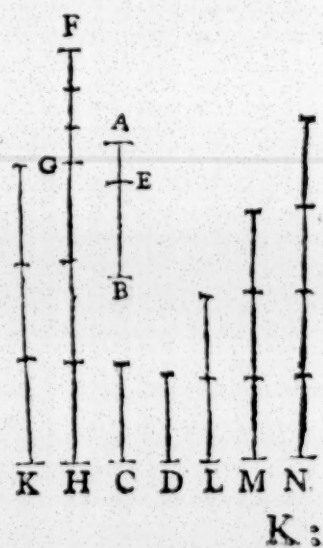
LET there be four magnitudes A, B, C, D ; and let E, F be any equimultiples of A and C the first and third : and G and H any other equimultiples of B and D the second and fourth : And let any equimultiples of A and C ; and any equimultiples of B and D be so particularly connected with one another ; that whenever the multiple of A exceeds the multiple of B ; the multiple of C will always exceed the multiple of D ; and when equal, equal : and when less, less : I say, that if any equimultiples of E and F be taken ; and any whatever of G and H ; it will always happen that when the multiple of E exceeds the multiple of G ; the multiple of F will exceed the multiple of H ; and when equal, equal ; and when less, less. That is take K and L any equimultiples whatever of E and F ; and also M and N any equimultiples whatever of G and H ; I say that when K exceeds M ; L will always exceed N ; and when equal, equal ; and when less, less.

THE

THE DEMONSTRATION. Because E and F are equimultiples of A and C; and K and L equimultiples of E and F; therefore, by the third proposition of the fifth book, K and L are equimultiples of A and C: Again because G and H are equimultiples of B and D; and M and N of G and H; therefore (by 3. 5.) M and N are equimultiples of B and D: but *the supposition is*, that if there be any equimultiples of A and C; and any whatever of B and D; if the multiple of A exceed the multiple of B; than the multiple of C will always exceed the multiple of D; and when equal, equal: and when less, less; therefore because it has been demonstrated that K and L are equimultiples of A and C and M and N of B and D; it follows that when K exceeds M; L will exceed N; and when equal, equal: and when less; less. Which was to be demonstrated.



Let AB, C and D be three such straight lines; that any equimultiples whatever of AB and C compared with any multiple of D; may be thus related; that whenever the multiple of AB exceeds the multiple of D; the multiple of C shall also exceed the multiple of D; and when equal, equal: and when less, less: I say if this be the case AB is equal to C. For if not, one of them must be greater, which suppose to be AB. Cut off a part from AB equal to C; and let this be BE: multiply AE their difference until it exceed D: and let this multiple be FG; therefore FG is greater than D: make GH the same multiple of EB; that FG is of AE; therefore (by 1. 5.) FH is also the same multiple of AB: make K the same multiple of C; therefore FH and K are equimultiples of AB and C. Make L the double of D, and so on, until we come to the multiple which is the next greater than K: let this be N: M therefore is not greater than



K : but K is equal to GH ; because EB is equal to C ; therefore M is not greater than GH ; but FG is greater than D (by const.) wherefore FH is greater than M and D together, that is N ; therefore FH is greater than N ; and K is not greater than N : and FH and K are equimultiples of AB and C and N is a multiple of D ; but by the supposition any equimultiples of AB and C were so related to any multiple of D ; that whenever the multiple of AB exceeds the multiple of D ; the multiple of C also exceeds the multiple of D ; but the contrary has been proved when we suppose AB unequal to C ; therefore the multiples being according to the supposition AB cannot be unequal to C. Which was to be demonstrated.

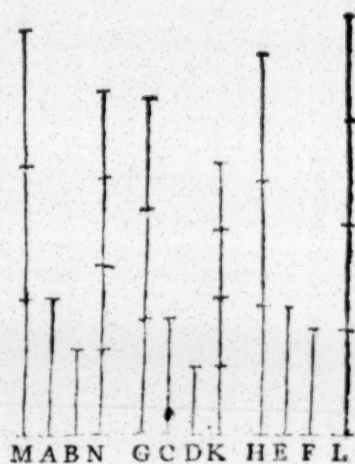
Let there be six magnitudes, or rather straight lines A, B, C, D, E, F whose multiples are thus related : suppose any equimultiples of A and C to be taken, and also any equimultiples whatever of B and D ; and farther whenever the multiple of A exceeds the multiple of B ; let the multiple of C always exceed the multiple of D : and again let there be some equimultiples of C and E and also of D and F such that the multiple of C exceeds the multiple of D ; but the multiple of E does not exceed the multiple of F : I say that there are some equimultiples of A and E ; and some of B and F such that the multiple of A exceeds the multiple of B ; when the multiple of E does not exceed the multiple of F.

Suppose G and H to be such equimultiples of C and E ; and K and L such equimultiples of D and F ; that G exceeds K but H does not exceed L : now because by the supposition G and K are given, I can find what number of times C and D are contained in them by measuring them, that is by cutting off parts equal to C and D. Take M the same multiple of A that G is of C ; that is, which H is of E : again take N the same multiple of B which K is of D ; that is, which L is of F.

DEMONSTRATION. Now by the supposition A, B, C, D are so related to one another that any equimultiples of A and C ; and also any equimultiples of B and D being taken ; if the multiple of A exceed the multiple of B the multiple of C exceeds the multiple of D : but M and G are (by const.) equimultiples of A and C ; as also N and K of B and D ; therefore if M exceed N ; G exceeds

exceeds K : but G does exceed K (by supp.); therefore M exceeds N : but H does not exceed L (by supp.); wherefore M exceeds N but H does not exceed L ; and M and H are equimultiples of A and E ; and N and L of B and F. Wherefore &c. Which was to be demonstrated.

If D and E (see the next figure) be such equimultiples of A and B ; and F such a multiple of C, that D exceeds F but E does not exceed F ; then I say that A is greater than B.



For because D exceeds F and E does not exceed F ; therefore of consequence D exceeds E : but because D and E are equimultiples of A and B and D exceeds E, therefore any part of D will exceed the same part of E ; that is if one of the magnitudes be greater than another, the third &c. part of the one will exceed the third &c. part of the other ; therefore A is greater than B.

Again if F be such a multiple of C ; and D and E such equimultiples of A and B, that F exceeds E but F does not exceed D ; then, I say that B is less than A.



For, because F exceeds E but F does not exceed D ; therefore E is less than D ; therefore any part of E will be less than the same part of D ; but A and B are the same parts of D and E therefore B is less than A. Which was to be demonstrated.

CHAP. VI.

Containing an explanation of the remaining definitions.

IF what has been said, in this dissertation, be properly attended to, there will be little difficulty in understanding the meaning of the third definition ; which has been the occasion of so much trouble to the commentators ; and which they have been so very unsuccessful

unsuccessful in their attempts to explain. For by proceeding according to the specimen here given, it is very obvious that all the properties of magnitudes, which are mentioned in the fifth book, may be demonstrated from the two first definitions, without any mention of the term ratio. But to save the trouble of such circumlocutions; our author gives particular names to the different suppositions; and this is the true origin of the following definitions, from the second. Now says he, in the third, any certain mutual habitude of magnitudes of the same kind, considered according to this *quantuplicity* or multiplicity; or according to this relation of parts and multiples, I call *ratio*. And I believe I may say that this definition has never been rightly understood since the greek became a dead language. *Barrow* calls it a metaphysical definition; but it appears from this, that it is a mathematical one. He says that such definition was equally necessary in the seventh book, when speaking of the ratio of numbers, but it appears from this explanation, that such a definition could have no place there.

When any equimultiples of the first and third, and any equimultiples whatever of the second and fourth, have this relation to one another, that when the multiple of the first exceeds the multiple of the second; the multiple of the third always exceeds the multiple of the fourth; or when equal, equal; or when less, less: then he says the first has the same ratio to the second which the third has to the fourth: But if the multiple of the first exceed the multiple of the second, when the multiple of the third does not exceed the multiple of the fourth; he then says that the first has a greater ratio to the second than the third has to the fourth. The sixth, eighth, and ninth definitions are sufficiently obvious. When three magnitudes are proportionals; the first is said to have to the third the duplicate ratio, of that which it has to the second. But this definition and the next I shall have occasion to explain afterwards.

When we say that the first has the same ratio to the second which the third has to the fourth; the first and third are the antecedents; and the second and fourth are called the consequents. Now the antecedents are called homologous terms, or terms of like ratio; and so are the consequents also. This distinction is necessary for
several

several reasons ; in the following definitions, and in fixing the equal angles of similar triangles &c, and therefore ought to be particularly attended to.

The following definitions will be sufficiently understood, when the particular propositions are read, wherein it is proved that the magnitudes will have the same ratio, the first to the second and the third to the fourth, after they have undergone the changes, mentioned in these definitions. I know there is a way of explaining these definitions by numbers, which is nothing to the purpose ; for Euclid is not speaking of numbers at present but of magnitudes. However as this supplies the place of all the knowledge which this book contains to many, who are not so scrupulous as to require a demonstration ; nor so attentive as to consider whether they are talking about something or nothing, to gratify such indolent readers, I have presented them with the following scheme, served up in the true French taste.

15 : 5 :: 30 : 10	Alt.	15 : 30 :: 5 : 10	By equal. ord.	{ 3 . 6 . 18 5 . 10 . 30
	Inv.	5 : 15 :: 10 : 30		
	Comp.	{ 15 + 5 : 5 :: 30 + 10 : 10 20 : 5 :: 40 : 10	By equal. pert.	{ 7 . 21 . 42 8 . 16 . 48
	Div.	{ 15 - 5 : 5 :: 30 - 10 : 10 10 : 5 :: 20 : 10		
	Conv.	15 : 15 - 5 :: 30 : 30 - 10 15 : 10 :: 30 : 20		
				3 : 18 :: 5 : 30 7 : 42 :: 8 : 48

I hope, from what has been said, the reader understands the meaning of these definitions. The triangle, rectangle and circle have been considered as instruments of investigation ; but this doctrine of *ratios* is to be regarded as a new mode of comparison, very extensive, in its consequences, giving us a wonderful command over the magnitudes, which we have already considered, by discovering a great many properties far beyond the reach of the former method of comparison. And although a great deal of pains has been taken to render our ideas upon this subject, confused, indistinct and nothing ; yet it is wonderful, when properly examined, what a simplicity there is in the principles upon which this method of comparison is founded ; the constructions are all performed (by 3. 1.) and requires no such *apparatus* as is necessary even

to prove one line to be equal to, or shorter than another, when we cannot make use of the circle.

C H A P. VII.

Of the arrangement of the propositions.

THE reader who has been accustomed to have his head filled with numerical ideas as explanatory of the propositions contained in this book, will perhaps be surprized to hear me affirm that he has taken a great deal of pains to fill his head with absurdities. For I am persuaded that if a scheme were to be thought of, for depriving a man of his reason, still keeping up an impression upon his mind that he was a rational creature, there could not be a more effectual method, than to set him upon reconciling the demonstrations of this book to numerical ideas. Indeed the twisting ropes of sand would be a rational employment when compared with this.

All numerical reasoning proceeds upon the supposition, that the unit is the same. But Euclid is not yet prepared for this confinement, which I shall prove particularly, upon the seventh book. He does not carry on his demonstrations in the first four books, upon the supposition that the sides of his triangles, or parallelograms or the radius of his circle, are three or four feet long; or as having a reference to any kind of measure. Nor is his reasoning in this book less general. His *ratio* is a mutual habitude of magnitudes according to quantuplicity, or according to the doctrine of parts and multiples. The question is not; is the part *one*; but is it a fixt magnitude? and farther, if you mean to reason upon it according to this doctrine of *ratios*; is it a magnitude that you can multiply? that is, that you can double; for the whole operation consists only of doubling. It would not be more absurd, to suppose that the nature of things, or human imagination does not allow of any triangle but an equilateral one, and nevertheless to try to prove that the square of one side of a triangle might be equal to the squares of the other two; than to attempt the proof of some of the simplest properties of triangles, parallelograms and circles upon the supposition that all magnitudes can be express'd by numbers.

It

It is to be hoped what has been said will be sufficient to induce the reader to pay a proper attention to the structure of the demonstrations made use of in this book, which have a wonderful elegance and force even when taken singly; but more particularly when the arrangement of the whole is considered: for the better understanding of which I would advise the reader to divide the book into four portions. The nature of *ratios* shews the propriety and necessity of introducing the properties of equimultiples first of all; and although the fourth proposition is not absolutely confined to its place; yet as it depends upon the third, it would be difficult to give it a more advantageous position. The first six propositions therefore will make the first portion. Again, we may have a very distinct notion of this *ratio* of magnitudes without knowing what the magnitudes themselves are directly: Here therefore some rule or test is necessary for enabling us to get at the common relations of greater, equal and less, when we have only this *ratio* to guide us. The student is therefore to consider the seventh, eighth, ninth and tenth propositions, as making a distinct section, and as introduced for the very purpose of settling this point.

It might appear to a superficial observer, who had just come from proving that if four magnitudes are proportionals, they will be proportionals by inversion; that a similar method of reasoning might be applied to shew that they will be proportionals by alternation. But it happens here, that the equimultiples, of the first and third, of the magnitudes which we want to prove to be proportionals; are not equimultiples of the second and fourth of those magnitudes which are proportionals by the supposition; but of the first and second; which destroys all prospect of succeeding according to that plan. And indeed this demonstration is no easy matter, for all the propositions from the tenth to the seventeenth may be considered as introduced to demonstrate; that magnitudes will be proportionals by alternation. And this will make up the third portion according to the division above mentioned. And the remaining part of the book may be considered under one head, in which is proved that magnitudes will be proportionals; by composition division, conversion, and by the two methods of reasoning by *equal distances*. And thus I think a very distinct and comprehensive view

of the subject of this book may be taken, and such an one as seems likely to fix it in the memory.

CHAP. VIII.

Containing some remarks on the demonstrations:

AGREEABLE to the rule already laid down, instead of remarks upon particular demonstrations, I shall reduce what I have to say under distinct heads, which the reader may apply as he goes along; but first it seems necessary to be a little particular, as to the construction of the thirteenth proposition. Indeed there is nothing which seems to be so prevailing an error, with regard to this book, as a neglect of the constructions. And yet this is a thing which Euclid is, every where, more particularly attentive to. He sometimes leaves his reader to draw the consequence from a construction, when it is sufficiently obvious, and cannot be done shortly and elegantly; an instance of which we have in the first proposition of the third book; where there are two constructions, but he reasons directly, only from one of them, and then joins both their consequences in his conclusion: for the demonstration only proves that the center must be in the perpendicular; for no absurdity will follow from the reasoning, by supposing the point G to be any where in the line CD ; not even if we suppose it to be at the circumference; for it will not then be absurd to suppose the angle GDB equal to CDB ; we have therefore Euclid's authority for drawing consequences in this manner; and I have not the least doubt of this demonstration being the genuine one given by Euclid; because to have made it quite regular the two constructions must have been separated; and then it would not have been one, but two distinct propositions; in the first of which it was to be proved that the center must be in the perpendicular; and in the second, that this perpendicular cut in halves would find the center; which is a much greater formality than the nature of the problem requires, although the student ought to examine it in this manner. But in this thirteenth proposition the case seems to be very different, and consequences are drawn from a construction which I believe

believe it will be found very difficult to perform. *Simson* in his note upon this proposition says; "In Commandine's, Briggs's and Gregory's translations, at the beginning of this demonstration, "it is said, *and the multiple of C is greater than the multiple of D; but the multiple of E is not greater than the multiple of F; which words are a literal translation from the greek: But the sense evidently requires that it be read, so that the multiple of C be greater than the multiple of D; but the multiple of E be not greater than the multiple of F.*" Now I am so much of a contrary opinion, that it appears to me that this change destroys every vestige of the true construction of this proposition. My notion is that the proper equimultiples are given me by the hypothesis; and the multiples and magnitudes being given, I can measure the one by the other; and consequently know how many times the magnitudes are taken; from which the construction is obvious: but if I have nothing to direct me but the magnitudes themselves; concerning which also there is no supposition of greater or less, I should be glad to know in such circumstances, how this construction is to be performed; I know it is said *let them be taken*, which if I were to add my conjectures after the manner of some commentators, I would suspect to be the correction of some hasty editor who did not understand the demonstration.

In proving four magnitudes to have the same ratio, the first to the second which the third has to the fourth; it is necessary that the equimultiples of the first and third be *any whatever*; and also that the equimultiples of the second and fourth be taken at a venture. Now it is obvious that if they be taken as often as one determinate magnitude contains another, they cannot be *any whatever*; neither is the sum of two multiples, each of which has been taken at a venture, to be considered as any multiple whatever; because if in one case it should happen to be taken three times, and in the other five times; these multiples added together, would not *happen* to be eight times that magnitude, but must really be eight times the magnitude, not accidentally but consequentially: again when a multiple of a magnitude is taken, and then a multiple of that multiple as in the third proposition; this last is not *any multiple whatever* of the first magnitude; for instance the first happens

to be taken three times, and this multiple again five times ; now the last multiple does not *happen* to be fifteen times the first, magnitude, but must be really so. And this will be sufficient to direct the student to the proper use of the phrase *any whatever*.

Simson's remarks upon the eighteenth proposition are very ingenious ; for the possibility of the fourth proportional is not to be taken for granted : because this is a very different notion from that which superficial people are apt to confound with it : for when two magnitudes are made the subject of contemplation, their *existence* is a part of the supposition ; and if they do exist, it must be under one or other of these forms ; A must be equal to B, or greater, or less ; and if I prove that A cannot be greater than B nor less, I certainly demonstrate that A is equal to B. But it is very different where the *existence* of the magnitude may be called in question, which is very reasonably done in the present instance ; for it may be said to Euclid you have confined yourself so much by your definition of *the same ratio*, that a fourth proportional may be really impossible according to your definition ; but when you have once shewn the possibility of it, I will then admit your reasoning. It is very certain, that we have sufficient principles, before this proposition, for finding a fourth proportional to any three given straight lines, which would make the demonstration unexceptionable as we have it at present ; by proceeding thus ; to AB, BE and CD find a fourth proportional, which must be greater, equal or less than FD &c ; but this would be mixing things together, which our author certainly intended to keep separate. But to conclude this chapter, whoever has a proper notion of a part and multiple ; and knows what is required to prove that four magnitudes have the same ratio the first to the second which the third has to the fourth, can never be mistaken in forming a judgement of the demonstrations contained in this book. Euclid constantly supposes his magnitudes to be straight lines as is obvious from his constructions and demonstrations ; so that there is no occasion to suppose the magnitudes in any proposition to be of the same kind ; and indeed those critics who have made such alterations, if they had understood their business, ought at least to have changed the enunciations of half the propositions in the book.

C H A P.

CHAP. IX.

Of the subject and arrangement of the sixth book.

OUR author having laid down this general doctrine of parts and multiples proceeds to apply it to the investigation of such properties of the triangle, rectangle and circle as could not be obtained by the former method of comparison. For what is delivered in the fifth book itself cannot, strictly speaking, be considered as properties of triangles, parallelograms &c. but only as properties of parts and multiples of such magnitudes as come under the two first definitions. But in this book he treats of the triangle as such, and shews when triangles have the same altitude, they have the same ratio to one another, as their bases. And here the thoughtless reader will do well to observe; that every triangle may have its altitude express'd by three different straight lines, as each side can be the base: for the altitude of a triangle is a relative term, and has no meaning until the base is fixt. From this he demonstrates that if the two sides of a triangle be cut by a straight line parallel to the third side, they will be cut in the same proportion; this is the fundamental principle which runs through the whole of this book; and ought for this reason to be examined in every point of view. For after the reader has considered this property according to its most obvious acceptation, he will find it more general than he suspected, by making it a property of two indefinite straight lines intersecting one another, and cut in any manner by two parallel lines.

It has been mentioned already that the triangle is the great instrument of geometrical investigation, and Euclid's first object in this book is to lay down those principles upon which the similarity of triangles depends; and by this acquisition the power of the science becomes astonishingly great, as it is difficult to say what cannot be done by it that is really practicable; the judicious reader therefore will examine the first eight propositions with that attention which the importance of the subject requires. The five following propositions teach how to divide a straight line into any number of equal parts; to cut a given line in the same proportion,
as

as another has been cut ; to find a third, fourth and mean proportional ; which being practical principles, should be made habitual. The next four propositions are all upon the same subject, and that a very curious one, namely to fix the equality of the parallelograms and triangles from the ratio of the sides and conversly. And here the indolent reader, should be required to shew what use is made of the equality of the angles, or where the demonstration would fail if that circumstance were omitted. The nineteenth and twentieth propositions are of great importance and should be carefully studied : it will assist a beginner very much to consider them as problems, omitting the term duplicate ratio entirely : Thus the nineteenth may be expressed in this manner ; two similar triangles being given, to find two straight lines which shall have the same ratio to one another which the triangles have. And BC and BG will be the two straight lines required. The reader may know whether he understands the proposition, by taking a third proportional to EF and BC, and completing the construction and demonstration. The twentieth may be worded thus ; to find two straight lines which shall have the same ratio to one another which two similar polygons have ; and then to prove that the same two straight lines, will have the same ratio to one another which the similar triangles, into which the polygons are divided, have to one another. And it will also be convenient, according to the same plan, to consider the twenty third as a problem ; which may be expressed in this manner ; to find two straight lines which shall have the same ratio to one another which two equiangular parallelograms have ; without making any use of the phrase *ratio compounded of ratios* ; and for a proper explanation of this term the reader may consult *Simson's* edition. I would propose it to the reader to examine these propositions first as problems, because this will fix his attention more to the particular steps of the construction and the consequences drawn from them. And also the extent and importance of these very beautiful theorems is more to be seen by considering them in this practical point of view. For by the twentieth proposition, figures may be increased or diminished in any proportion. For instance if I want to make a figure the fifth part of a given figure ; I take the fifth part of one of the sides of the given figure ; and find a
mean

mean proportional between the line and its fifth part, upon which if a similar and similarly situated figure be described, it will be the fifth part of the given figure. Likewise by this proposition the forty seventh of the first book, may be extended to any similar rectilineal figures.

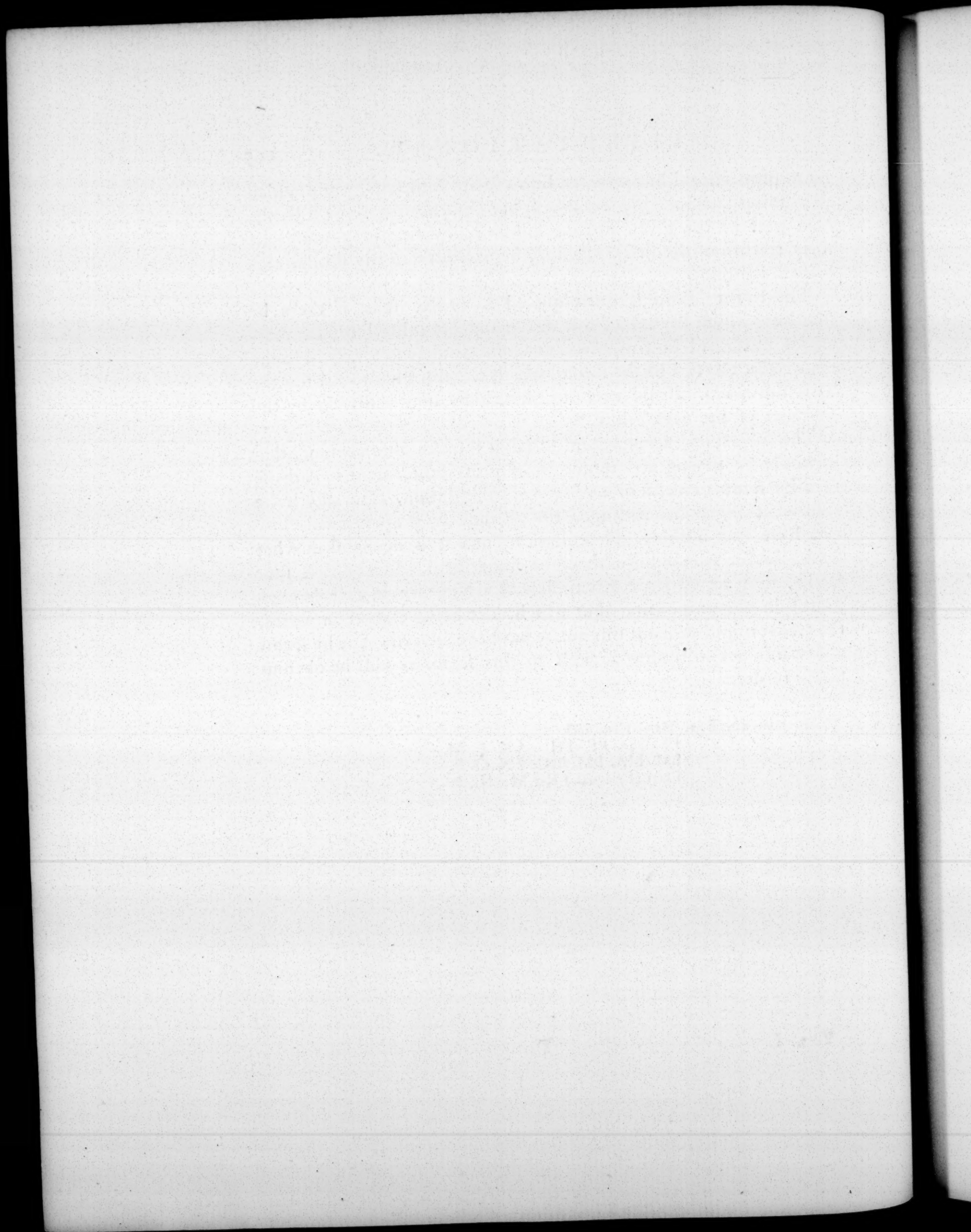
The twenty third is often quoted to prove, that, if there be four lines which have the same ratio; and also other four, the rectangles contained by the antecedents and consequents will have the same ratio, the first to the second which the third has to the fourth. This the reader should examine, which he may easily do by the assistance of the note below. *

The subject and arrangement of what remains is so obvious, that it would be paying the reader but a poor complement to suppose that he stood in need of any farther information, especially if he read what *Simson* has said upon the 28, and 29 propositions.

I have said nothing of corollaries; and it is doubtful in what sense Euclid would have them be understood. In the first four books, they are all some circumstances which are in fact demonstrated in the proposition, but which, not being expressed in words, the reader might without this notice have overlooked; but he seems afterwards not to confine himself to this sense, as will be obvious to the reader.

* See prop. 10. B. 1. Ham. Con. Sect.

EP. PD. PF. PH	{	K. L. M.
BM. MA. BM. MC.		N. O. P.
Then (by equal.) K : M :: N : P.		



T H E
E L E M E N T S
O F
E U C L I D.
B O O K V.

D E F I N I T I O N S.

1. **A** Part is a magnitude of a magnitude, the less of the greater, ^{Book V.} when it measures the greater. 2. But a multiple, the greater of the less, when it is measured by the less.

3. Ratio is a certain habitude of magnitudes of the same kind to one another, which *is only* as to quantity.

4. Magnitudes are said to have a ratio to one another, which may be multiplied so as to exceed one another.

5. Magnitudes are said to be in the same ratio ; the first to the second and the third to the fourth ; when the equimultiples of the first and third, according to any multiplication, are at the same time less, or at the same time equal, or are at the same time greater than each of the equimultiples of the second and fourth, compared with one another. 6. And let magnitudes, having the same ratio, be called proportionals. 7. But when of equimultiples ; the multiple of

Book V. the first exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth; then the first is said to have to the second a greater ratio than the third to the fourth. 8. And proportion is a similitude of ratios. 9. But proportion consists in three terms at least. 10. And when three magnitudes are proportionals; the first is said to have to the third a duplicate ratio of that which it has to the second. 11. And when four magnitudes are proportionals; the first is said to have to the fourth a triplicate ratio of that which it has to the second; and always in order one more, as long as the proportion continues.

12. The leading magnitudes are said to be of like ratio with the leading magnitudes; and those that follow *are said to be of like ratio* with those that follow. OR. The antecedents are said to be homologous magnitudes to the antecedents; and the consequents to the consequents.

13. Alternate ratio is the taking of the antecedent to the antecedent; and of the consequent to the consequent.

14. Inverse ratio is the taking of the consequent as an antecedent to the antecedent as a consequent.

15. The composition of a ratio is the taking of the antecedent together with the consequent, as one, to the consequent. 16. But the division of a ratio is the taking of the excess, by which the antecedent exceeds the consequent, to the consequent itself.

17. The conversion of a ratio is the taking of the antecedent, to the excess, by which the antecedent exceeds the consequent.

18. A ratio of equality is, there being several magnitudes, and others equal to them in multitude, and in the same proportion taken two by two, when it is, as in the first magnitudes; the first to the last, so in the second magnitudes the first to the last. or otherwise. The taking of the extremes by a subtraction of the middle *terms*.

19. Ordinate proportion is, when it is as the antecedent to the consequent, so is the antecedent to the consequent; and it is as the consequent to some other, so is the consequent to some other.

20. But perturbate proportion is, when there being three magnitudes and others equal to them in number, it is as in the first magnitudes the antecedent to the consequent, so in the second magnitudes

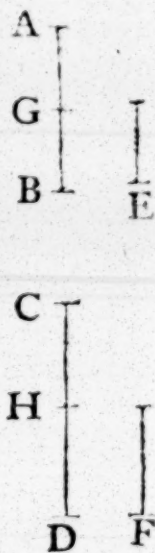
tudes the antecedent to the consequent ; but as in the first magnitudes the consequent is to some other, so in the second magnitudes is some other to the antecedent. Book V.

P R O P. I.

If there be any number of magnitudes, equimultiples of an equal number of magnitudes, each of each ; whatsoever multiple one of the magnitudes is of one, the same multiple will all be of all.

Let AB, CD be any number of magnitudes, equimultiples of an equal number of magnitudes E, F ; each of each : I say that whatsoever multiple AB is of E, the same multiple will AB, CD together be of E, F together.

For because AB is the same multiple of E that CD is of F ; as many magnitudes as there are in AB equal to E, so many are there in CD equal to F ; let AB be divided (by 3. 1.) into the magnitudes AG, GB equal to E ; and CD into the magnitudes CH, HD equal to F : certainly the number of parts CH, HD will be equal to the number of parts AG, GB : and because AG is equal to E, and CH equal to F, therefore AG, CH together are equal to E and F together ; for the same reason GB is equal to E, and HD to F ; therefore also GB, HD together are equal to E, F together : therefore as many magnitudes as there are in AB equal to E ; so many are there in AB, CD together equal to E, F together : wherefore whatsoever multiple AB is of E, the same multiple will AB, CD together be of E, F together.



Wherefore if there be any number of magnitudes, equimultiples of an equal number of magnitudes, each of each ; whatsoever multiple one of the magnitudes is of one, the same multiple will all be of all. Which was to be demonstrated.

P R O P. II.

If the first be the same multiple of the second as the third is of the fourth ; and if the fifth be the same multiple of the second as

Book V. the sixth *is* of the fourth : also the first and fifth together will be the same multiple of the second, as the third and sixth together *is* of the fourth.

For let the first AB be the same multiple of the second C as the third DE is of the fourth F ; and let the fifth BG be the same multiple of the second C as the sixth EH is of the fourth F ; I say that AG the first together with the fifth will be the same multiple of the second C as DH the third and sixth *is* of F the fourth.

For because AB is the same multiple of C that DE is of F ; as many magnitudes as there are in AB equal to C, so many *will there be* in DE equal to F : certainly for the same reason also, as many *magnitudes* as there are in BG equal to C, so many are there also in EH equal to F ; therefore as many as there are in the whole AG equal to C so many *are there* in the whole DH equal to F : wherefore whatsoever multiple AG is of C, the same multiple will DH be of F ; therefore AG the first together with the fifth will be the same multiple of C the second, that DH the third and sixth *is* of F the fourth.



Wherefore if the first be the same multiple of the second as the third *is* of the fourth ; and if the fifth be the same multiple of the second as the sixth *is* of the fourth ; also the first and fifth together will be the same multiple of the second, as the third and sixth together *is* of the fourth. Which was to be demonstrated.

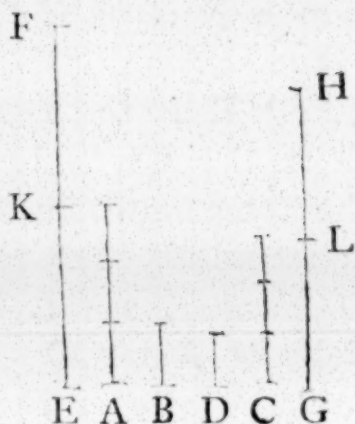
P R O P. III.

If the first be the same multiple of the second as the third *is* of the fourth ; and equimultiples be taken of the first and third ; also by equality, each of those taken will be an equimultiple of each, the one of the second, and the other of the fourth.

For let the first A be the same multiple of the second B, as the third C *is* of the fourth D ; and let EF, GH be taken equimultiples of A and C : I say that EF is the same multiple of B, which GH *is* of D.

For

For because EF is the same multiple of A (by const.) which GH is of C : therefore as many *magnitudes* as there are in EF equal to A, so many *are there* in GH equal to C : let EF be divided into *magnitudes* equal to A, viz. EK, KF ; and GH into *magnitudes* equal to C, viz. GL, LH ; the number of *parts* EK, KF will be equal to the number of *parts* GL, LH : And because A is the same multiple of B, as C is of D ; and EK is equal to A, and GL equal to C ; there-



fore EK is the same multiple of B, which GL is of D. Certainly for the same reason also KF is the same multiple of B, which LH is of D : wherefore because the first EK is the same multiple of the second B, as the third GL is of the fourth D ; and also the fifth KF is the same multiple of the second B, which the sixth LH is of the fourth D ; therefore (by 2. 5.) EF the first together with the fifth is the same multiple of the second B, which GH the third and sixth is of the fourth D.

Wherefore if the first be the same multiple of the second, as the third is of the fourth ; and equimultiples be taken of the first and third : also by equality, each of those taken will be an equimultiple of each, the one of the second, and the other of the fourth. Which was to be demonstrated.

P R O P. IV.

If the first have the same ratio to the second as the third to the fourth ; also the equimultiples of the first and third, will have the same ratio to the equimultiples of the second and fourth, according to any multiplication whatsoever ; taken so as to answer each other.

For let the first A have the same ratio to the second B, which the third C has to the fourth D ; and let E and F be taken equimultiples of A and C ; but let G and H be taken any equimultiples of B and D which may accidentally happen ; I say that as E is to G so is F to H.

For let K and L be taken equimultiples of E and F ; and M, N any other equimultiples of G and H which may happen.

And

Book V. And because E is the same multiple of A which F is of C; and K and L have been taken equimultiples of E and F; therefore (by 3. 5.) K is the same multiple of A which L is of C; Certainly for the same reason M is the same multiple of B which N is of D; and because (by supp.) A is to B as C is to D; and K and L have been taken equimultiples of A and C; and M and N [any other] equimultiples of B and D [which may accidentally happen]; therefore (by 5 def. 5.) if K exceed M also L exceeds N, and if equal, equal; and if less, less: and K and L are equimultiples of E and F; and M and N any other equimultiples, which may accidentally happen; of G and H; therefore (by 5 def. 5.) E is to G as F is to H.



Wherefore if the first have the same ratio to the second as the third to the fourth; also the equimultiples of the first and third, will have the same ratio to the equimultiples of the second and fourth; according to any multiplication whatsoever; taken so as to answer each other. Which was to be demonstrated.

Cor. Wherefore because it has been demonstrated, that if K exceed M; L exceeds N, and if equal, equal: and if less, less. Certainly also if M exceed K, N exceeds L: and if equal, equal; and if less, less: and on this account it will be; as G is to E, so is H to F. Certainly from this it is manifest that if four magnitudes be proportionals, they will be inversly proportionals; (by def. 14. 5.).

PROP. V.

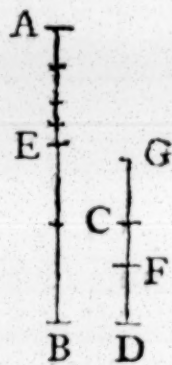
If a magnitude be the same multiple of a magnitude, which a *magnitude* taken from *the one* is of a *magnitude* taken from *the other*; whatsoever multiple the whole is of the whole, also the remainder will be the same multiple of the remainder.

For let the magnitude AB be the same multiple of the magnitude CD, which AE the *magnitude* taken from *the one* is of the *magnitude*

magnitude CF taken from *the other* : I say that whatsoever multiple Book V.
the whole AB is of the whole CD, also EB the remainder will be {
the same multiple of FD the remainder,

For whatsoever multiple AE is of CF ; let EB be the same multiple of CG.

And because AE is the same multiple of CF (by the const. and 1. 5.) *which* AB is of GF ; and AE is supposed *to be* the same multiple of CF *which* AB is of CD : therefore AB is the same multiple of either of the *lines* GF, CD ; therefore (by com. not. 7.) GF is equal to CD ; let CF *which is* common be taken away ; therefore the remainder GC is equal to the remainder DF ; and because AE is the same multiple of CF which EB is of GC ; and GC is equal to DF ; therefore AE is the same multiple of CF which EB is of FD ; but AE is supposed *to be* the same multiple of CF, which AB is of CD : therefore EB is the same multiple of FD *which* AB is of CD ; therefore whatsoever multiple the whole AB is of the whole CD ; also the remainder EB is the same multiple of the remainder FD.



Wherefore if a magnitude be the same multiple of a magnitude, which a *magnitude* taken from *the one* is of a magnitude taken from *the other* ; whatsoever multiple the whole is of the whole, also the remainder will be the same multiple of the remainder. Which was to be demonstrated.

P R O P. VI.

If two magnitudes be equimultiples of two magnitudes ; and some *magnitudes* equimultiples of the same be taken away ; also the remainders are either equal to those magnitudes, or equimultiples of them.

For let the two magnitudes AB, CD be equimultiples of the two magnitudes E, F ; and let AG, CH the *magnitudes* taken away be equimultiples of the same E, F : I say that the remainders GB, HD are either equal to E, F or equimultiples of them.

For first let GB be equal to E ; I say that HD is also equal to F : for put CK equal to F.

And

Book V.

And because AG is the same multiple of E as CH is of F; and GB is equal to E; and CK equal to F; therefore (by 2. 5.) AB is the same multiple of E which KH is of F; but AB is supposed to be the same multiple of E, which CD is of F; therefore KH is the same multiple of F, which CD is of F; wherefore because each of the lines KH, CD is the same multiple of F; therefore (by com. not. 6.) KH is equal to CD; let CH which is common be taken away; therefore the remainder KC is equal to the remainder HD: but KC is equal to F; therefore HD is also equal to F; so that when GB is equal to E, HD will also be equal to F.



Certainly in the same manner we shall demonstrate, that when GB is a multiple of E, that HD will be the same multiple of F.

Wherefore if two magnitudes be equimultiples of two magnitudes; and some magnitudes equimultiples of the same be taken away; also the remainders are either equal to those magnitudes, or equimultiples of them. Which was to be demonstrated.

P R O P. VII.

Equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to equal magnitudes.

Let A and B be equal magnitudes; and C any other magnitude which may accidentally happen; I say that each of the magnitudes A, B has the same ratio to C; and that C has the same ratio to each of the magnitudes A, B.

For let D, E be taken equimultiples of A, B; and let F be taken any other multiple of C which may accidentally happen.

And because D is the same multiple of A, which E is of B; and A is (by supp.) equal to B; therefore D is equal to E; but F is any other multiple of C which may accidentally happen; wherefore if D exceed F, E also exceeds F; and if equal, equal; and if less, less: and D, E are equimultiples of A, B; and F any other multiple of C which may accidentally happen; therefore (by

(by 5 def. 5.) it is as A is to C so is B to C.

I say also that C has the same ratio to each of the *magnitudes* A, B,

For the same things being constructed ; in the same manner we shall demonstrate that D is equal to E ; and that F is any other *magnitude* ; therefore if F exceed D ; it also exceeds E : and if equal, equal : and if less, less : and F is a multiple of C ; and D, E any other equimultiples of A and B which may accidentally happen ; therefore (by 5. def. 5.) it is, as C to A so is C to B.



Therefore equal *magnitudes* have the same ratio to the same *magnitude* ; and the same *magnitude* has the same ratio to equal *magnitudes*. Which was to be demonstrated.

P R O P. VIII.

Of unequal magnitudes ; the greater has a greater ratio to the same *magnitude*, than the less *has* : and the same *magnitude* has a greater ratio to the less, than it has to the greater.

Let AB and C be unequal magnitudes ; and let AB be greater than C ; and let D be any other magnitude which may accidentally happen : I say that AB has a greater ratio to D than C has to D : And D has a greater ratio to C, than it *has* to AB.

For because AB is greater than C ; make BE equal to C : certainly (by 4. def. 5.) the lesser of the *two* AE, EB being multiplied will at length be greater than D. First let AE be less than EB ; and let AE be multiplied until what is produced shall be greater than D : and let FG be the multiple of it, which is greater than D : and whatsoever multiple FG is of AE, let GH be the same multiple of EB ; and K *the same multiple* of C : and let L be taken the double of D, and M triple ; and more by one in order, until the multiple of D taken becomes the first greater than K : let it be taken : and let it be N four times D, the first greater than K.

Wherefore because K is the first less than N or [N is the first multiple of D greater than K] ; therefore K is not less than M ;

Book V. and because FG is the same multiple of AE

which GH is of EB; therefore (by 1. 5.) FG is the same multiple of AE which FH is of AB; and FG is the same multiple of AE which K is of C; therefore FH is the same multiple of AB, which K is of C; therefore FH and K are equimultiples of AB and C: Again because GH is the same multiple of EB which K is of C; and EB is equal to C; therefore GH is equal to K: but K is not less than M; therefore neither is GH less than M; but (by const.) FG is greater

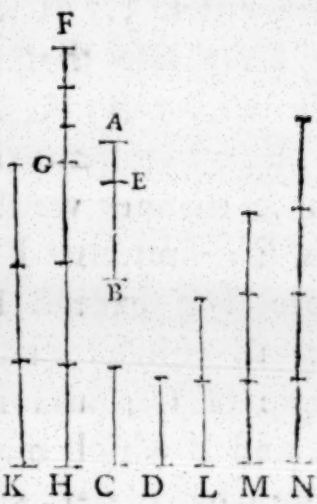
than D; therefore the whole FH is greater than both D and M together; but D and M together are equal to N; therefore FH exceeds N, but K does not exceed N; and FH and K are equimultiples of AB and C; and N is any other multiple of D which may accidentally happen; therefore (by 7. def. 5.) AB has a greater ratio to D, than C has to D.

I say also, that D has a greater ratio to C, than D has to AB.

For the same things being constructed; in like manner we shall demonstrate, that N exceeds K; but does not exceed FH; and N is a multiple of D; and FH and K [any other] equimultiples of AB and C [which may accidentally happen]; therefore (by 7. def. 5.) D has a greater ratio to C, than D has to AB.

But let AE be greater than EB; certainly EB the less being multiplied will at length (by 4. def. 5.) be greater than D: let it be multiplied; and let GH be the multiple of EB, greater than D; and whatsoever multiple GH is of EB; let FG be made the same multiple of AE; and K of C: Certainly in the same manner we shall demonstrate, that FH and K are equimultiples of AB and C: and in like manner let N be taken, a multiple of D, the first greater than FG; so that again FG be not less than M; but GH is greater than D; therefore the whole FH exceeds

D and M together, that is N; but K does not exceed N; since FG being



being greater than GH; that is K, does not exceed N; and in like manner following the *steps* above we finish the demonstration. Book V.

Wherefore of unequal magnitudes, the greater has a greater ratio to the same *magnitude*, than the less *has*: and the same *magnitude* has a greater ratio to the less, than it has to the greater. Which was to be demonstrated.

P R O P. IX.

Magnitudes having the same ratio to the same *magnitude* are equal to one another; and those are also equal to one another, to which the same *magnitude* has the same ratio.

For let each of the *magnitudes* A, B have the same ratio to the same *magnitude* C: I say that A is equal to B.

For if not, (by 8. 5.) each of the *magnitudes* A, B could not have the same ratio to C: but it has (by supp.); therefore A is equal to B.

Again let C have the same ratio to each of the *magnitudes* A, B; I say that A is equal to B.

For if not; C could not (by 8. 5.) have the same ratio to each of the *magnitudes* A, B; but it has (by supp.); therefore A is equal to B.

Wherefore, *magnitudes* having the same ratio to the same *magnitude* are equal to one another; and those are equal to one another to which the same *magnitude* has the same ratio. Which was to be demonstrated.

P R O P. X.

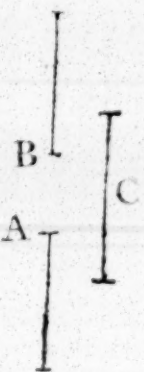
Of *magnitudes* having ratio to the same *magnitude*, that is the greater which has the greater ratio: and that is the less to which the same has a greater ratio.

For let A have a greater ratio to C than B has to C: I say that A is greater than B.

For if not; either A is equal to B, or less; but A is not equal to B; for *then* each of the *magnitudes* A, B (by 7. 5.) has the same ratio to C; but (by supp.) *each of them* has not: therefore A is not equal to B; neither is A less than B; for (by 8. 5.) A would

* B 2

have



Book V. have a less ratio to C than B *would have* to C; but (by
 ~~~~~ sup.) it has not; therefore A is not less than B; but it  
 has been demonstrated that neither is it equal: therefore  
 A is greater than B.

Again let C have a greater ratio to B than C has to A:  
 I say that B is less than A.

For if not; it is either equal, or greater: but B is not  
 equal to A, for (by 7. 5.) C would have the same ratio  
 to each of the *magnitudes* A, B: but (by sup.) it has not;  
 therefore A is not equal to B: Neither is B greater than A; for  
 (by 8. 5.) C would have a less ratio to B than to A; but (by sup.)  
 it has not; therefore B is not greater than A; but it has been de-  
 monstrated that neither *is* it equal; therefore B is less than A.

Wherefore of *magnitudes* having ratio to the same *magnitude*,  
 that is the greater, which has the greater ratio: and that is the  
 less to which the same has a greater ratio. Which was to be de-  
 monstrated.

### P R O P. XI.

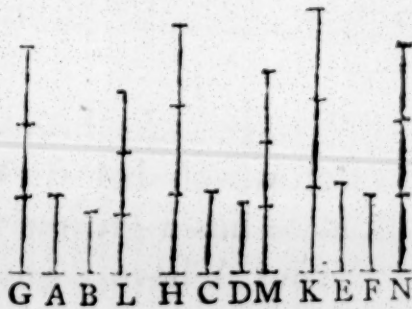
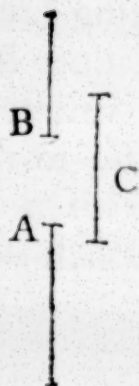
*Ratios* which are the same to the same ratio, are the same to one  
 another.

For let the *ratios* be; as A to B so is C to D; and as C to D  
 so is E to F; I say that it is, as A is to B so is E to F.

For let G, H, K be taken equimultiples of A, C, E; and let  
 L, M, N be taken any other equimultiples of B, D, F which may  
 accidentally happen.

And because it is (by sup.) as A is to  
 B so is C to D; and G and H have been  
 taken equimultiples of A and C; and L,  
 M any equimultiples of B, D which may  
 accidentally happen; wherefore if G ex-  
 ceed L; H also exceeds M (by 5. def. 5.);  
 and if equal, equal; and if less, less:

Again, because it is (by sup.), as C is to D so is E to F; and H,  
 K have been taken equimultiples of C, E; and M, N any other  
 equimultiples of D, F which may accidentally happen; wherefore  
 (by 5. def. 5.) if H exceed M; K also exceeds N; and if equal,  
 equal;





equal; and if less, less; but if H exceed M; G also exceeds L; Book V.  
 and if equal, equal; and if less, less: and G, K are equimultiples  
 of A, E; and L, N any other equimultiples of B, F which may  
 accidentally happen; wherefore (by 5. def. 5.) it is, as A is to B  
 so is E to F.

Wherefore *ratios* which are the same to the same ratio; are the  
 same to one another. Which was to be demonstrated.

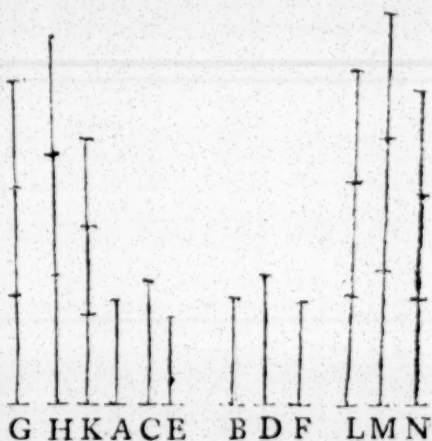
P R O P. XII.

If any number of magnitudes be proportionals; it will be, as  
 one of the antecedents is to one of the consequents, so are all the  
 antecedents to all the consequents.

Let A, B, C; D, E, F be any number of magnitudes, propor-  
 tionals; viz. as A is to B so let C be to D; and E to F: I say that  
 it is as A is to B so is A, C, E to B, D, F.

For let G, H, K equimultiples of A, C, E be taken; and L, M,  
 N any other which may accidentally happen of B, D, F.

And because it is, as A is to B so is C  
 to D; and E to F (by supp.); and G,  
 H, K have been taken equimultiples of  
 A, C, E; and L, M, N any other equi-  
 multiples of B, D, F which may acci-  
 dentally happen; wherefore (by 5. def.  
 5.) if G exceed L; H also exceeds M  
 and K exceeds N; and if equal, equal;  
 and if less, less: so that also if G exceed  
 L; G, H, K taken together also exceeds



L, M, N taken together; and if it be equal; they are equal; and  
 if it be less, they are less; and G; and G, H, K are equimultiples  
 of A, and of A, C, E; since (by 1. 5.) if there be any number of  
 magnitudes equimultiples of an equal number of magnitudes, each  
 of each; whatsoever multiple any one magnitude is of one; the  
 same multiple will, all be of all: Certainly for the same reason  
 also; L, and L, M, N are equimultiples of B, and of B, D, F:  
 therefore (by 5. def. 5.) it is, as A is to B so is A, C, E to B, D, F.

Wherefore if any number of magnitudes be proportionals; it  
 will be as one of the antecedents is to one of the consequents, so  
 are



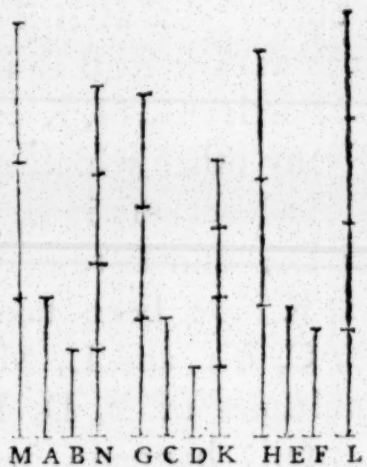
Book V. are all the antecedents to all the consequents. Which was to be demonstrated.

## PROP. XIII.

If the first have the same ratio to the second which the third has to the fourth : but if the third have a greater ratio to the fourth than the fifth *has* to the sixth ; the first also will have a greater ratio to the second than the fifth *has* to the sixth.

For let the first A have the same ratio to the second B, *which* the third C has to the fourth D : but let the third C have a greater ratio to the fourth D, than the fifth E, has to the sixth F ; I say that the first A will have a greater ratio to the second B, than the fifth E has to the sixth F.

For because C has a greater ratio to D (by supp.), than E has to F ; (by 7. def. 5.) there are some equimultiples of C and E ; and other equimultiples [which may accidentally happen] of D and F : and the multiple of C exceeds the multiple of D : but the multiple of E does not exceed the multiple of F : let them be taken ; and let G, H be the equimultiples of C, E ; and K, L other equimultiples [which may accidentally happen] of D, F ; so that G exceeds K ; but H does not exceed L : and whatsoever multiple G is of C, let M be the same multiple of A : and whatsoever multiple K is of D, let N be the same multiple of B.



And because it is, as A *is* to B so *is* C to D ; and M *and* G have been taken equimultiples of A, C ; and N, K other equimultiples [which may accidentally happen] of B, D : wherefore (by 5. def. 5.) if M exceed N, G also exceeds K : and if equal, equal : and if less, less : But G exceeds K (by supp.) therefore also M exceeds N : but H does not exceed L ; and M, H are equimultiples of A, E ; and N, L other equimultiples [which may accidentally happen] of B, F ; therefore (by 7. def. 5.) A has a greater ratio to B, than E *has* to F,

Where-



Wherefore if the first have the same ratio to the second which the third has to the fourth; but *if* the third have a greater ratio to the fourth than the fifth *has* to the sixth; the first also will have a greater ratio to the second than the fifth *has* to the sixth. Which was to be demonstrated. Book V.

P R O P. XIV.

If the first have the same ratio to the second, *which* the third *has* to the fourth; and if the first be greater than the third; also the second will be greater than the fourth; and if equal, equal: and if less, less.

For let the first A have the same ratio to the second B, *which* the third C *has* to the fourth D; and let A be greater than C: I say that B is greater than D.

For because (by supp.) A is greater than C; and B is any other magnitude which may happen: therefore (by 8. 5.) A has a greater ratio to B, than C *has* to B; but as A *is* to B (by supp.) so *is* C to D; therefore (by 13. 5.) C has a greater ratio to D, than C *has* to B: but (by 10. 5.) that *magnitude* is the less, to which the same has a greater ratio; therefore D is less than B; so that B is greater than D.



Certainly in the same manner we shall demonstrate, than when A is equal to C, B will also be equal to D: and if A be less than C, B will be less than D.

Wherefore if the first have the same ratio to the second, which the third *has* to the fourth; and if the first be greater than the third; also the second will be greater than the fourth; and if equal, equal; and if less, less. Which was to be demonstrated.

P R O P. XV.

The parts, compared with one another, have the same ratio *which* their equimultiples *have*.

For let AB be the same multiple of C *which* DE *is* of F: I say that it is, as C *is* to F so *is* AB to DE.

For



Book V.

For because (by supp.) AB is the same multiple of C which DE is of F; therefore as many magnitudes as there are in AB equal to C, so many is there also in DE equal to F; let AB be divided into magnitudes equal to C, viz. AG, GH, HB; and DE into magnitudes equal to F, viz. DK, KL, LE: certainly the number of parts AG, GH, HB will be equal to the number of parts DK, KL, LE: and because AG, GH, HB are equal to one another; also DK, KL, LE are equal to one another; therefore it is, (by 7.

5.) as AG is to DK so is GH to KL, and HB to LE; wherefore (by 12. 5.) it will be as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; wherefore it is, as AG is to DK so is AB to DE; but AG is equal to C, and DK is equal to F; therefore it is, as C is to F so is AB to DE.

Wherefore the parts, compared with one another, have the same ratio which their equimultiples have. Which was to be demonstrated.

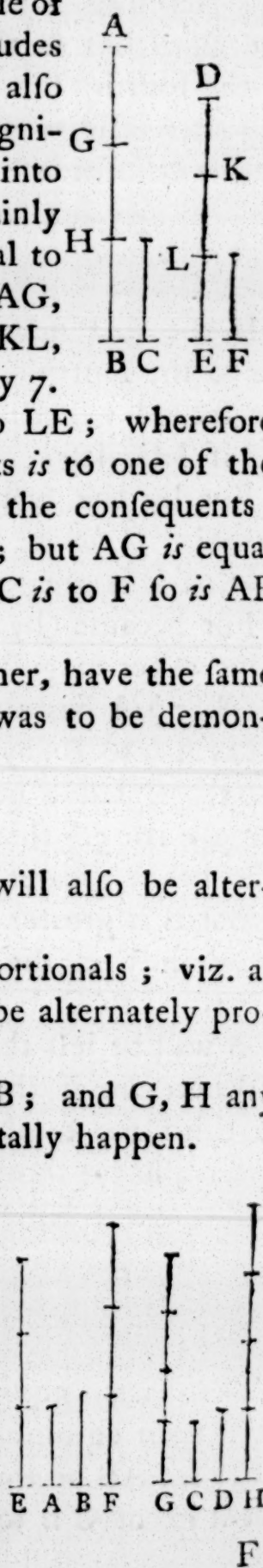
## P R O P. XVI.

If four magnitudes be proportionals; they will also be alternately proportionals.

Let the four magnitudes, A, B, C, D be proportionals; viz. as A is to B so let C be to D: I say that they will be alternately proportionals: viz. as A is to C so will B be to D.

For let E, F be taken equimultiples of A, B; and G, H any other equimultiples of C, D which may accidentally happen.

And because E is the same multiple of A which F is of B; and (by 15. 5.) parts, compared with one another have the same ratio which their equimultiples have: therefore it is, as A is to B so is E to F; but (by supp.) as A is to B so is C to D; and therefore as C is to D so (by 11. 5.) is E to F: Again, because G and H are equimultiples of C and D; therefore (by 15. 5.) C is to D as G is to H; but as C is to D so is E to



F;



F ; therefore (by 11. 5.) as E is to F so is G to H ; but if four <sup>Book V.</sup> magnitudes be proportionals, and the first be greater than the third ; the second (by 14. 5.) will be greater than the fourth : and if equal, equal : and if less, less : wherefore if E exceed G, F also exceeds H : and if equal, equal : and if less, less : and E and F are equimultiples of A and B, and G and H any other equimultiples of C and D which may accidentally happen ; therefore (by 5. def. 5.) as A is to C so is B to D.

Wherefore if four magnitudes be proportionals they will be (by 13. def. 5.) alternately proportionals. Which was to be demonstrated.

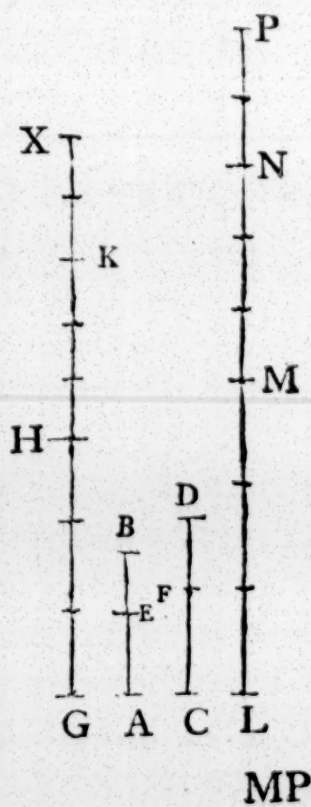
P R O P. XVII.

If compounded magnitudes be proportionals ; they will also be proportionals *when* divided.

Let the compounded magnitudes, AB, BE, CD, DF be proportionals ; viz. as AB is to BE so let CD be to DF : I say that they will be proportionals when divided ; that is (by 16. def. 5.) as AE is to EB so will CF be to FD.

For let GH, HK, LM, MN be taken equimultiples of AE, EB, CF, FD : and let KX, NP be taken any other equimultiples of EB, FD which may accidentally happen.

And because GH is the same multiple of AE which HK is of EB ; therefore GH is the same multiple of AE which GK is of AB (by 1. 5.) ; but GH is the same multiple of AE which LM is of CF : therefore GK is the same multiple of AB which LM is of CF. Again because LM is the same multiple of CF which MN is of FD : therefore (by 1. 5.) LM is the same multiple of CF which LN is of CD : but LM was the same multiple of CF which GK is of AB : therefore GK is the same multiple of AB which LN is of CD : therefore GK, LN are equimultiples of AB, CD. Again because HK is the same multiple of EB which MN is of FD ; and KX is also the same multiple of EB which NP is of FD : also (by 2. 5.) HX is the same multiple of EB which





Book V. *MP is of FD : and because it is (by supp.) as AB is to BE so is CD to DF ; and GK, LN have been taken equimultiples of AB, CD ; and HX, MP [any other] equimultiples of EB, FD [which may accidentally happen] ; therefore (by 5. def. 5.) if GK exceed HX, LN also exceeds MP : and if equal, equal : and if less, less : Let GK exceed HX, and, the common part HK being taken away, therefore also GH exceeds KX : But if GK exceed HX, LN also exceeds MP ; therefore also LN exceeds MP ; and MN which is common being taken away ; LM also exceeds NP : so that if GH exceed KX ; LM also exceeds NP. Certainly in the same manner we shall demonstrate that if GH be equal to KX ; LM will also be equal to NP : and if less, less : But GH, LM are equimultiples of AE, CF ; and KX, NP any other equimultiples of EB, FD which may accidentally happen ; therefore (by 5. def. 5.) it is, as AE is to EB so is CF to FD.*

Wherefore if compounded magnitudes be proportionals ; they will also be proportionals *when* divided. Which was to be demonstrated.

#### P R O P. XVIII.

If magnitudes *when* divided are proportionals ; they will be proportionals *when* compounded.

Let, AE, EB, CF, FD the divided magnitudes be proportionals, viz. as AE is to EB so let CF be to FD : I say also that they will be proportionals *when* compounded, that is (by 15. def. 5,) as AB is to BE so will CD be to DF.

For if it is not, as AB is to BE so is CD to DF ; it will be as AB is to BE so is CD, either to some *magnitude* less than FD, or to one greater.

Let it be first to DG a less : and because it is (by this supp.) as AB is to BE so is CD to DG, the compounded magnitudes are proportionals : so that also (by 17. 5.) they will be proportionals when divided ; therefore it is, as AE is to EB so is CG to GD : but it is also supposed that as AE is to EB so is CF to FD : wherefore (by 11. 5.) as CG is to GD so is CF to FD : but CG the first is greater than CF the third ; therefore also (by 14. 5.) the second GD is greater





than the fourth FD ; but it is also less : which is impossible : Book V.  
wherefore it is not, as AB is to BE so is CD to a *magnitude* less {  
than DF : Certainly in the same manner we shall demonstrate,  
that neither is it to *one* greater. Therefore, it is to *that one* itself.

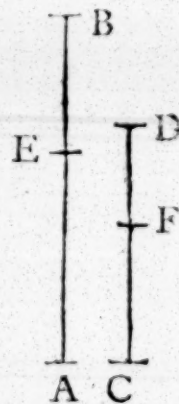
Wherefore if magnitudes *when* divided are proportionals ; they  
will be proportionals *when* compounded. Which was to be de-  
monstrated.

P R O P. XIX.

If it be, as the whole *magnitude* is to the whole, so is a *magni-  
tude* taken away to a *magnitude* taken away ; the remainder also will  
be to the remainder as the whole is to the whole.

For let it be, as the whole AB is to the whole CD, so let AE a  
*magnitude* taken away be to CF a *magnitude* taken away ; I say that  
EB the remainder will also be to FD the remainder, as the whole  
AB is to the whole CD.

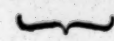
For since it is, as the whole AB is to the whole CD  
so is AE to CF : also alternately (by 16. 5.) as AB is  
to AE so is CD to CF ; and because the compounded  
magnitudes are proportionals, they will also be pro-  
portionals when divided (by 17. 5.) ; therefore as BE  
is to EA so is DF to FC ; and therefore alternately (by  
16. 5.), as BE is to DF so is EA to FC : but as EA is  
to FC so is the whole AB supposed to be to the whole  
CD ; therefore also (by 11. 5.) the remainder EB will  
be to the remainder FD as the whole AB is to the  
whole CD.



Wherefore if it be, as the whole *magnitude* is to the whole, so  
is a *magnitude* taken away to a *magnitude* taken away ; the remainder  
also will be to the remainder as the whole is to the whole. Which  
was to be demonstrated.

Cor. And because it has been demonstrated (in this prop.) as  
AB is to CD so is EB to FD : and alternately (by 16. 5.) as AB  
is to EB so is CD to FD : therefore the compounded magnitudes  
are proportionals : but it has been demonstrated (in this and prop.  
16. 5.) as AB is to AE so is CD to CF ; and *this* is the conversion of  
*the proportionals*. Certainly from this it is manifest, that if com-  
pounded



Book V.  pounded magnitudes be proportionals, they will be proportionals by conversion (by 17. def. 5.). Which was to be demonstrated.

## PROP. XX.

If there be three magnitudes and others, equal to them in multitude, in the same ratio, taken two and two; and by equality, if the first be greater than the third; also the fourth will be greater than the sixth: and if equal, equal: and if less, less.

Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude, in the same ratio taken two and two; viz. as A is to B so let D be to E, and as B is to C so let E be to F: and by equality, let A be greater than C; I say also that D will be greater than F: and if equal, equal: and if less, less.

For because A is greater than C, and B is any *magnitude* which may accidentally happen: but (by 8. 5.) the greater has a greater ratio to the same than the less *has*: therefore A has a greater ratio to B than C *has* to B: but (by supp.) as A is to B so is D to E: but by inversion (by cor. to 4. 5.) C is to B as F is to E: therefore (by 11. et 13. of 5.) D has a greater ratio to E than F has to E: but of *magnitudes* having ratio to the same; that is the greater which has the greater ratio (by 10. 5.): therefore D is greater than F: Certainly in the same manner we shall demonstrate, that if A be equal to C, D will also be equal to F: and if less, less.



Wherefore if there be three magnitudes, and others equal to them in multitude, in the same ratio, taken two and two; and by equality, if the first be greater than the third; also the fourth will be greater than the sixth; and if equal, equal: and if less, less. Which was to be demonstrated.

## PROP. XXI.

If there be three magnitudes, and others equal to them in multitude; and in the same ratio, taken two and two, and their proportion be perturbate; and if by equality the first be greater than the

the



the third, the fourth will also be greater than the sixth : and if Book V.  
equal, equal : and if less, less.

Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude, and in the same ratio, taken two and two ; and let their proportion be perturbate, viz. as A is to B so let E be to F ; and as B is to C so let D be to E : and by equality let A be greater than C : I say also that D will be greater than F : and if equal, equal : and if less, less.



For because A is greater than C, and B is some other *magnitude* : therefore (by 8. 5.) A has a greater ratio to B than C has to B : but as A is to B so (by supp.) is E to F : and as C is to B so by inversion is E to D : therefore (by 11. and 13. 5.) E has a greater ratio to F than E has to D : but (by 10. 5.) that is the less *magnitude* to which the same has a greater ratio ; therefore F is less than D : wherefore D is greater than F : Certainly we shall demonstrate in the same manner, that if equal, equal ; certainly if A be equal to C, D also will be equal to F : and if less, less.

Wherefore if there be three magnitudes and others equal to them in multitude ; and in the same ratio, taken two and two, and their proportion be perturbate ; and if by equality the first be greater than the third, the fourth will also be greater than the sixth : and if equal, equal : and if less, less. Which was to be demonstrated.

P R O P. XXII.

If there be any number of magnitudes ; and others equal to them in multitude ; in the same ratio, taken two and two ; also by equality they will be in the same ratio.

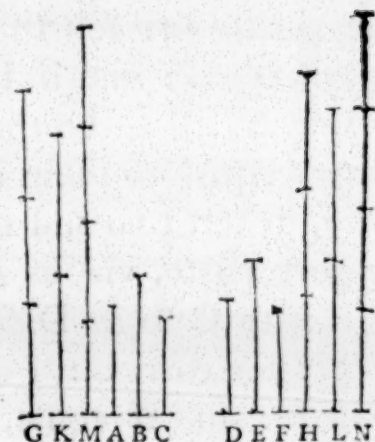
Let there be any number of magnitudes A, B, C ; and D, E, F others equal to them in multitude ; in the same ratio, taken two and two ; viz. as A is to B so let D be to E and as B is to C so let E be to F : I say also, that by equality they will be in the same ratio ; viz. (by 18. def. 5.) as A is to C so will D be to F.

For let G, H be taken equimultiples of A, D ; and K, L any other equimultiples of B, E which may accidentally happen ; and  
besides



Book V. besides, M, N any other equimultiples of  
 C, F which may accidentally happen.

And because it is (by supp.) as A is to B so is D to E; and G, H have been taken equimultiples of A, D; and K, L any other equimultiples of B, E which may accidentally happen: therefore (by 4. 5.) it is, as G is to K so is H to L. Certainly for the same reason also as K is to M so is L to N: wherefore because there are three magnitudes G, K, M; and others equal to them in multitude H, L, N and in the same ratio, taken two and two; therefore by equality, (by 20. 5.) if G exceed M, also H exceeds N: and if equal, equal: and if less, less: and G and H are equimultiples of A, D; and M, N any other equimultiples of C, F which may accidentally happen: therefore (by 5. def. 5.) it is, as A is to C so is D to F.



Wherefore if there be any number of magnitudes, and others equal to them in multitude; in the same ratio, taken two and two; also by equality, they will be in the same ratio. Which was to be demonstrated.

### P R O P. XXIII.

If there be three magnitudes, and others equal to them in multitude; in the same ratio, taken two and two; and if their proportion be perturbate; they will also be in the same ratio by equality.

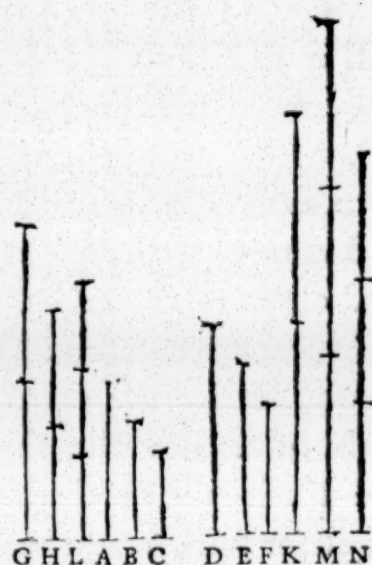
Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude; in the same ratio, taken two and two; and let their proportion be perturbate, viz. as A is to B so let E be to F; and as B is to C so let D be to E: I say that it is, as A is to C so is D to F.

For let G, H, K be taken equimultiples of A, B, D; and L, M, N any other equimultiples of C, E, F which may accidentally happen.

And because G, K are equimultiples of A, B; and parts, (by 15. 5.) have the same ratio to one another as their equimultiples; therefore it is, as A is to B so is G to H: Certainly for the same reason also as E is to F so is M to N: and because it is, as B is to C so



is D to E ; and H, K have been taken equimultiples of B, D ; and L, M any other equimultiples of C, E which may accidentally happen ; therefore it is, (by 4. 5.) as H is to L so is K to M : but it has been demonstrated as G is to H so is M to N : wherefore because there are three magnitudes G, H, L and others K, M, N equal to them in multitude ; in the same ratio taken two and two ; and their proportion is perturbate ; therefore by equality (by 21. 5.) if G exceed L ; K also exceeds N : and if equal, equal : and if less, less : and



G, K are equimultiples of A, D ; and L, N any other equimultiples of C, F which may accidentally happen ; therefore (by 5. def. 5.) A is to C as D is to F.

Wherefore if there be three magnitudes, and others equal to them in multitude ; in the same ratio taken two and two ; and if their proportion be perturbate ; they will also be in the same ratio by equality. Which was to be demonstrated.

P R O P. XXIV.

If the first have the same ratio to the second which the third has to the fourth ; and if the fifth have also the same ratio to the second which the sixth has to the fourth ; also the first and fifth together will have the same ratio to the second which the third and sixth together has to the fourth.

For let the first AB have the same ratio to the second C, which the third DE has to the fourth F : and also let the fifth BG have the same ratio to the second C, which the sixth EH has to the fourth F : I say that also AG the first together with the fifth will have the same ratio to C the second which DH the third together with the sixth has to F the fourth.

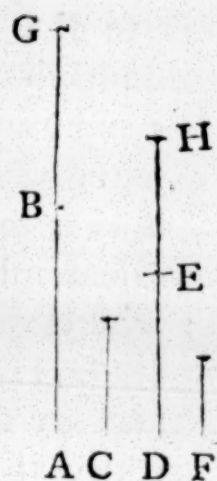
For because it is, as BG is to C so is EH to F : therefore by inversion (by cor. to 4. 5.) C is to BG as F is to EH, Wherefore because it is as AB is to C so is DE to F ; and as C is to BG so is F to



Book V.

F to EH: therefore by equality (by 22. 5.) it is, as AB is to BG so is DE to EH: and because the divided magnitudes are proportionals, they will also (by 18. 5) be proportionals *when* compounded: wherefore as AG is to GB so is DH to HE: but it is, as GB is to C so is HE to F: therefore by equality (by 22. 5.) it is, as AG is to C so is DH to F.

Wherefore if the first have the same ratio to the second *which* the third has to the fourth; and if the fifth have also the same ratio to the second *which* the sixth has to the fourth; also the first and fifth together will have the same ratio to the second *which* the third and sixth together has to the fourth. Which was to be demonstrated.



## PROP. XXV.

If four magnitudes be proportionals; the greatest and the least of them are greater than the other two.

Let the four magnitudes AB, CD, E, F be proportionals; viz. as AB is to CD so let E be to F: and let AB be the greatest of them and F the least; I say that AB and F are greater than CD and E.

For let AG be made equal to E and let CH be made equal to F.

Wherefore because it is, as AB is to CD so is E to F; and AG is equal to E and CH to F; therefore it is as AB is to CD so is AG to CH: and because it is as the whole AB is to the whole CD so is AG taken away to CH taken away; therefore (by 19. 5.) the remainder GB will be to the remainder HD as the whole AB



is to the whole CD: but (by supp.) AB is greater than CD; therefore GB is greater than HD: and because AG is equal to E and CH to F: therefore AG and F together are equal to CH and E together; and because if equals be added to unequals, the wholes are unequal: wherefore GB and HD being unequal: and AG and F be added to the greater GB: and CH and E be added to the less HD: AB and F together are greater than CD and E.

Wherefore if four magnitudes be proportionals, the greatest and the least of them are greater than the other two. Which was to be demonstrated.



THE  
ELEMENTS  
OF  
EUCLID.

BOOK VI.

DEFINITIONS.

1. **SIMILAR** rectilineal figures are those which have their <sup>Book VI.</sup> angles equal each to each; and the sides about the equal angles proportionals. 2. But those are reciprocal figures; when the antecedent and consequent terms are in each of the figures.
3. A straight line is said to have been cut in extreme and mean ratio, when it is, as the whole *line* is to the greater segment, so is the greater *segment* to the less.
4. The altitude of any figure is the perpendicular drawn from the vertex to the base.
5. A ratio is said to be compounded of ratios, when the quantities of the ratios being multiplied into one another do make some *ratio*.

VOL. I.

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PROP.



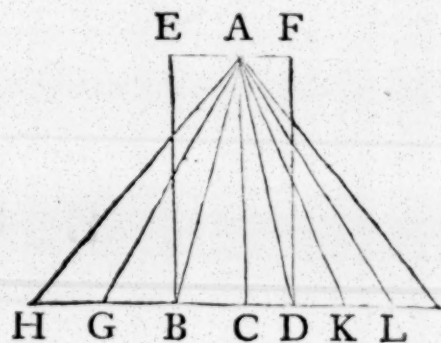
## PROP. I.

The triangles and parallelograms which are under the same altitude are to one another as *their* bases.

Let  $ABC, ACD$  be triangles; and  $EC, CF$  parallelograms, which are under the same altitude, viz. the perpendicular drawn from *the* point  $A$  to  $BD$ : I say that it is, as the base  $BC$  is to the base  $CD$  so is the triangle  $ABC$  to the triangle  $ACD$ : also so is the parallelogram  $EC$  to the parallelogram  $CF$ .

For let  $BD$  be produced towards both parts to the points  $H, L$ : and let any number of *lines*  $BG, GH$  be made equal to the base  $BC$ : and any number of *lines*  $DK, KL$  equal to the base  $CD$ : and let  $AG, AH, AK, AL$  be joined.

And because  $CB, BG, GH$  are equal to one another; also (by 38. 1.) the triangles  $AGH, AGB, ABC$  are equal to one another: therefore whatsoever multiple the base  $HC$  is of the base  $BC$ ; the same multiple also is the triangle  $AHC$  of the triangle  $ABC$ . Certainly for the same reason also, whatsoever multiple the base  $CL$  is of the base  $CD$  the same multiple is the triangle  $ACL$  of the triangle  $ACD$ . And if the base  $HC$  be equal to the base  $CL$  (by 38. 1.) the triangle  $AHC$  is also equal to the triangle  $ACL$ : and if the base  $HC$  exceed the base  $CL$ ; the triangle  $AHC$  also exceeds the triangle  $ACL$ : and if less, less. There being four magnitudes, the two bases  $BC, CD$ ; and the two triangles  $ABC, ACD$ : and equimultiples of the base  $BC$  and of the triangle  $ABC$  have been taken viz. the base  $HC$  and the triangle  $AHC$ : and of the base  $CD$  and of the triangle  $ACD$  any other equimultiples which may accidentally happen viz. the base  $CL$  and the triangle  $ACL$ : and it has been demonstrated that if the base  $HC$  exceed the base  $CL$ ; the triangle  $AHC$  also exceeds the triangle  $ACL$ : and if equal, equal: and if less, less: therefore (by 5. def. 5.) it is, as, the base  $BC$  is to the base  $CD$  so is the triangle  $ABC$  to the triangle  $ACD$ .



And



And because the parallelogram EC is double of the triangle ABC <sup>Book VI.</sup> (by 41. 1.): and the parallelogram FC is double of the triangle ACD; and parts (by 15. 5.) have the same ratio *to one another* as their equimultiples; therefore it is, as the triangle ABC is to the triangle ACD so is the parallelogram EC to the parallelogram FC: wherefore because it has been demonstrated *that* as the base BC is to the base CD so is the triangle ABC to the triangle ACD: and as the triangle ABC is to the triangle ACD so is the parallelogram EC to the parallelogram FC; therefore also (by 11. 5.) the base BC is to the base CD as the parallelogram EC is to the parallelogram FC.

Wherefore the triangles and parallelograms, which are under the same altitude, are to one another as *their* bases. Which was to be demonstrated.

## P R O P. II.

If any straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally: and if the sides of the triangle be cut proportionally, the straight line joining the sections will be parallel to the remaining side of the triangle.

For let DE be drawn parallel to BC one of the sides of the triangle ABC: I say that it is, as BD is to DA so is CE to EA.

For let BE, CD be joined.

Therefore the triangle BDE is equal to the triangle CDE; for they are upon the same base DE and between the same parallel lines DE, BC: but ADE is some other triangle; and (by 7. 5.) equal *magnitudes* have the same ratio to the same *magnitude*: therefore it is, as the triangle BDE is to the triangle ADE so is the triangle CDE to the triangle ADE: but as the triangle BDE is to the triangle ADE so is (by 1. 6.) BD to DA: for being under the same altitude, the perpendicular drawn from the *point* E to AB, they are to one another as their bases. Certainly for the same reason also, as the triangle CDE is to the triangle ADE so is CE to EA: Therefore (by 11. 5.) as BD is to DA so is CE to EA.

But let AB, AC, the sides of the triangle ABC be cut proportionally in the points D, E viz. as BD is to DA so let CE be to EA:

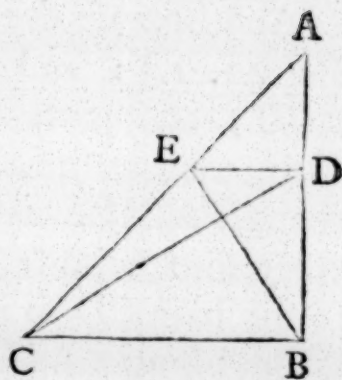
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and



Book VI. and let DE be joined: I say that DE is parallel to BC.

For the same things being constructed; because it is, as BD is to DA so is CE to EA: but as BD is to DA so (by 1. 6.) is the triangle BDE to the triangle ADE: and as CE is to EA so is the triangle CDE to the triangle ADE: therefore also (by 11. 5.) as the triangle BDE is to the triangle ADE so is the triangle CDE to the triangle ADE: therefore each of the triangles BDE, CDE has the same ratio to the triangle ADE; therefore (by 9. 5.) the triangle BDE is equal to the triangle CDE: and they are upon the same base DE: but equal triangles being upon the same base are also (by 39. 1.) between the same parallels: therefore DE is parallel to BC.



Wherefore if any straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally: and if the sides of the triangle be cut proportionally, the straight line joining the sections will be parallel to the remaining side of the triangle. Which was to be demonstrated.

### P R O P. III.

If an angle of a triangle be cut in halves, and the straight line cutting the angle also cut the base; the segments of the base will have the same ratio *which* the remaining sides of the triangle *have to one another*: and if the segments of the base have the same ratio *which* the remaining sides of the triangle *have to one another*; the straight line joining the vertex *and the point of section* cuts the angle of the triangle in halves.

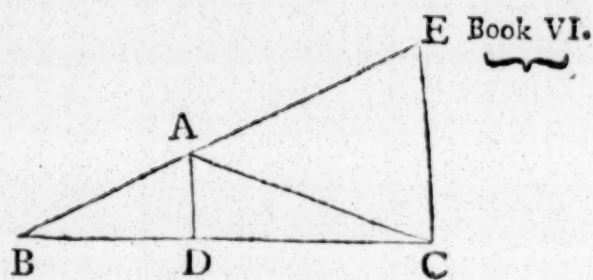
Let ABC be a triangle, and let the angle BAC be cut in halves by the straight line AD: I say that it is as BD is to DC so is BA to AC.

For let CE be drawn through the point C parallel to DA: and BA being produced let it meet CE in E.

And because the straight line AC hath fallen upon the parallels AD, EC; therefore the angle ACE (by 29. 1.) is equal to the angle



angle CAD : but the angle CAD is supposed equal to BAD : therefore the angle BAD is equal to the angle ACE. Again because the straight line BAE hath fallen upon the parallels AD, EC ; the outward angle BAD is equal (by 29. 1.) to the inward angle AEC : but



the angle ACE hath been also demonstrated to be equal to BAD ; therefore (by com. not. 1.) the angle ACE is equal to AEC : so that also (by 6. 1.) the side AE is equal to the side AC : and because AD has been drawn parallel to CE one of the sides of the triangle BCE ; there is *this* proportion therefore (by 2. 6.) as BD is to DC so is BA to AE : But AE is equal to AC : therefore it is (by 7. 5.) as BD is to DC so is BA to AC.

But let BD be to DC as BA to AC ; and let AD be joined : I say that the angle BAC hath been cut in halves by the straight line AD.

For the same things being constructed ; because it is (by supp.), as BD is to DC so is BA to AC : but also (by 2. 6.) as BD is to DC so is BA to AE ; for AD hath been drawn parallel to CE one of the sides of the triangle BCE : and therefore (by 11. 5.) as BA is to AC so is BA to AE ; therefore (by 9. 5.) AC is equal to AE ; so that also (by 5. 1.) the angle AEC is equal to the angle ACE : but the angle AEC is equal to the outward angle BAD (by 29. 1.) ; and also the angle ACE is equal to the alternate angle CAD ; therefore the angle BAD is equal to CAD : wherefore the angle BAC hath been cut in halves by the straight line AD.

Wherefore if an angle of a triangle be cut in halves ; and the straight line cutting the angle also cut the base ; the segments of the base will have the same ratio *which* the remaining sides of the triangle have to one another : and if the segments of the base have the same ratio *which* the remaining sides of the triangle have to one another ; the straight line, joining the vertex and the point of section, cuts the angle of the triangle in halves. Which was to be demonstrated.

P R O P.

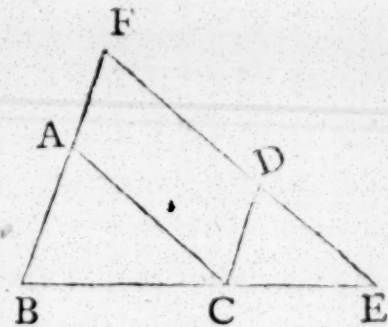


The sides about the equal angles of equiangular triangles are proportionals; and those sides are of like ratio *which are extended under the equal angles.*

Let  $ABC$ ,  $DCE$  be equiangular triangles, having the angle  $ABC$  equal to the angle  $DCE$ ; and the angle  $ACB$  equal to the angle  $DEC$ ; and besides the angle  $BAC$  equal to the angle  $CDE$ : I say that the sides about the equal angles of the triangles  $ABC$ ,  $DCE$  are proportionals; and *that* those sides are of like ratio *which are extended under the equal angles.*

For let  $BC$  be placed in a straight line with  $CE$ : and because the angles  $ABC$ ,  $ACB$  are less than two right angles; and  $ACB$  is equal to the angle  $DEC$ ; therefore the angles  $ABC$ ,  $DEC$  are less than two right angles; therefore (by *com. not.* 11.)  $BA$ ,  $ED$  produced will meet; let them be produced and meet in  $F$ .

And because the angle  $DCE$  is equal to the angle  $ABC$ ; therefore (by 28. 1.)  $BF$  is parallel to  $CD$ : Again, because the angle  $ACB$  is equal to  $DEC$ ,  $AC$  is parallel to  $FE$ ; therefore  $FACD$  is a parallelogram; therefore (by 34. 1.)  $FA$  is equal to  $CD$ ; and  $AC$  to  $FD$ : and because  $AC$  hath been drawn parallel to  $FE$  one of the sides of the triangle  $FBE$ ; therefore it is, (by 2.



6.) as  $BA$  is to  $AF$  so is  $BC$  to  $CE$ : but  $AF$  is equal to  $CD$ ; therefore as  $BA$  is  $CD$  (by 7. 5.) so is  $BC$  to  $CE$ ; and alternately (by 16. 5.) as  $AB$  is to  $BC$  so is  $DC$  to  $CE$ : Again, because  $CD$  is parallel to  $BF$ ; therefore (by 2. 6.) it is, as  $BC$  is to  $CE$  so is  $FD$  to  $DE$ : but  $FD$  is equal to  $AC$ : therefore as  $BC$  is to  $CE$  so is  $AC$  to  $DE$ ; therefore alternately, as  $BC$  is to  $CA$  so is  $CE$  to  $ED$ : wherefore because it has been demonstrated *that* as  $AB$  is to  $BC$  so is  $DC$  to  $CE$ ; and as  $BC$  is to  $CA$  so is  $CE$  to  $ED$ ; therefore by equality (by 22. 5.) *it is*, as  $BA$  is to  $AC$  so is  $CD$  to  $DE$ .

Wherefore the sides about the equal angles of equiangular triangles are proportionals; and the sides of like ratio are extended under the equal angles. Which was to be demonstrated.

PROP.



## PROP. V.

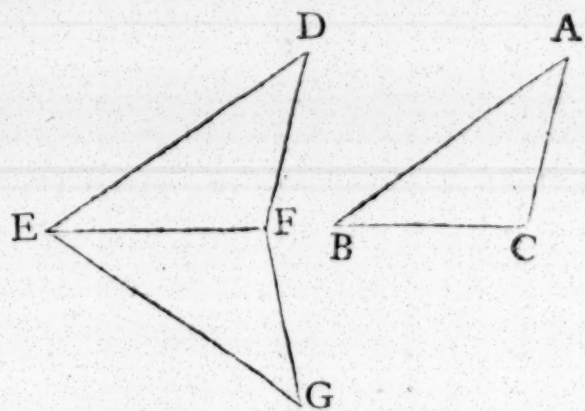
Book VI.

If two triangles have their sides proportionals, the triangles will be equiangular; and will have those angles equal under which the sides of like ratio are extended.

Let  $ABC$ ,  $DEF$  be two triangles having their sides proportionals, viz. as  $AB$  is to  $BC$  so let  $DE$  be to  $EF$ ; and as  $BC$  is to  $CA$  so let  $EF$  be to  $FD$ ; and besides as  $BA$  is to  $AC$  so let  $ED$  be to  $DF$ : I say, that the triangle  $ABC$  is equiangular to the triangle  $DEF$ , and will have those angles equal, under which the sides of like ratio are extended viz.  $ABC$  equal to  $DEF$ ; and  $BCA$  equal to  $EFD$ ; and besides  $BAC$  equal to  $EDF$ .

For with the straight line  $EF$  and at the points  $E$ ,  $F$  in it, let the angle  $FEG$  be made equal to  $ABC$ ; and let  $EFG$  be made equal to  $BCA$ ; therefore (by 32. 1.) the remaining angle  $BAC$  is equal to the remaining angle  $EGF$ .

Wherefore the triangle  $ABC$  is equiangular to the triangle  $EGF$ ; therefore the sides about the equal angles of the triangles  $ABC$ ,  $EGF$  are proportionals (by 4. 6.); and the sides of like ratio are extended under the equal angles; therefore it is, as  $AB$  is to  $BC$  so is  $GE$  to  $EF$ : but



as  $AB$  is to  $BC$  so is  $DE$  supposed to be to  $EF$ ; wherefore (by 11. 5.) as  $DE$  is to  $EF$  so is  $GE$  to  $EF$ : therefore each of the lines  $DE$ ,  $GE$  has the same ratio to  $EF$ ; therefore (by 9. 5.)  $DE$  is equal to  $GE$ . Certainly for the same reason also  $DF$  is equal to  $GF$ : wherefore because  $DE$  is equal to  $EG$  and  $EF$  common, certainly the two  $DE$ ,  $EF$  are equal to the two  $GE$ ,  $EF$ ; and the base  $DF$  is equal to the base  $GF$ ; therefore (by 8. 1.) the angle  $DEF$  is equal to the angle  $GEF$ ; and the triangle  $DEF$  is equal to the triangle  $GEF$ ; and the remaining angles are equal to the remaining angles, under which the equal sides are extended (by 4. 1.): therefore the angle  $DFE$  is equal to the angle  $GFE$ ; and  $EDF$  to  $EGF$ :



Book VI.  $\text{EGF}$  : and because  $\text{DEF}$  is equal to  $\text{GEF}$  : but  $\text{GEF}$  (by const.) is equal to  $\text{ABC}$  ; therefore also the angle  $\text{ABC}$  is equal to  $\text{DEF}$  : Certainly for the same reason also  $\text{ACB}$  is equal to  $\text{DFE}$  ; and besides (by 32. 1.) the angle at  $\text{A}$  is equal to the angle at  $\text{D}$  : Therefore the triangle  $\text{ABC}$  is equiangular to the triangle  $\text{DEF}$ .

Wherefore if two triangles have their sides proportionals, the triangles will be equiangular : and will have the angles equal, under which the sides of like ratio are extended. Which was to be demonstrated.

### PROP. VI.

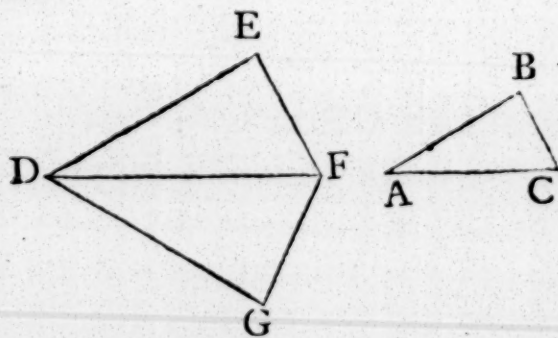
If two triangles have one angle equal to one angle, and the sides about the equal angles proportionals : the triangles will be equiangular, and will have the angles equal, under which the sides of like ratio are extended.

Let  $\text{ABC}$ ,  $\text{DEF}$  be two triangles, having one angle  $\text{BAC}$  equal to one angle  $\text{EDF}$  ; and the sides about the equal angles proportionals viz. as  $\text{BA}$  is to  $\text{AC}$  so let  $\text{ED}$  be to  $\text{DF}$  : I say that the triangle  $\text{ABC}$  is equiangular to the triangle  $\text{DEF}$  ; and will have the angle  $\text{ABC}$  equal to the angle  $\text{DEF}$  ; and  $\text{ACB}$  equal to  $\text{DFE}$ .

For with the straight line  $\text{DF}$  and at the points,  $\text{D}$ ,  $\text{F}$  in it ; let the angle  $\text{FDG}$  be made equal to either of the angles  $\text{BAC}$ ,  $\text{EDF}$  ; and let  $\text{DFG}$  be made equal to  $\text{ACB}$ .

Therefore the remaining angle at  $\text{B}$  is equal (by 32. 1.) to the remaining angle at  $\text{G}$  : therefore the triangle  $\text{ABC}$  is equiangular to the triangle  $\text{DGF}$  ; therefore there is this proportion (by 4. 6.) as  $\text{BA}$  is to  $\text{AC}$  so is  $\text{GD}$  to  $\text{DF}$  ; but

it is also supposed that as  $\text{BA}$  is to  $\text{AC}$  so is  $\text{ED}$  to  $\text{DF}$  ; therefore also (by 11. 5.) as  $\text{ED}$  is to  $\text{DF}$  so is  $\text{GD}$  to  $\text{DF}$  ; therefore (by 9. 5.)  $\text{ED}$  is equal to  $\text{DG}$  ; and  $\text{DF}$  is common : certainly the two  $\text{ED}$ ,  $\text{DF}$  are equal to the two  $\text{GD}$ ,  $\text{DF}$  ; and (by const.) the angle  $\text{EDF}$  is equal to the angle  $\text{GDF}$  : therefore (by 4. 1.) the base  $\text{EF}$  is equal to the base  $\text{FG}$  ; and the triangle  $\text{DEF}$  is equal to the triangle





angle DGF; and the remaining angles will be equal to the re-<sup>Book VI.</sup>maining angles, each to each, under which the equal sides are extended; therefore the angle DFG is equal to the angle DFE; and the *angle* at G to the *angle* at E; but the angle DFG is equal (by const.) to the angle ACB; therefore also ACB is equal to DFE: but BAC is also supposed equal to EDF; therefore the remaining *angle* at B is equal to the remaining *angle* at E: wherefore the triangle ABC is equiangular to the triangle DEF.

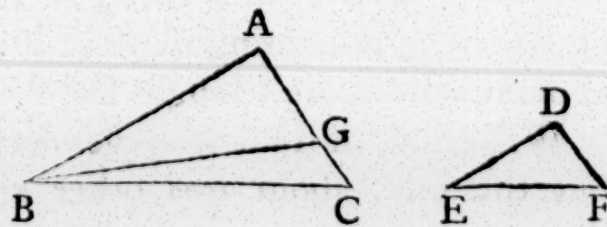
Wherefore if two triangles have one angle equal to one angle, and the sides about the equal angles proportionals: the triangles will be equiangular, and will have the angles equal, under which the sides of like ratio are extended. Which was to be demonstrated.

## P R O P. VII.

If two triangles have one angle equal to one angle; and the sides about *two* other angles proportionals; and if each of the remaining angles *be* at the same time less or not less than a right angle: the triangles will be equiangular; and will have those angles equal, about which the sides are, *which are* proportionals.

Let ABC, DEF be two triangles having one angle equal to one angle viz. the *angle* BAC equal to the *angle* EDF; and the sides about *two* other angles proportionals, the *angles* ABC, DEF; so that AB may be to BC as DE to EF; and first let each of the remaining angles at C and F be at the same time less than a right angle: I say, that the triangle ABC is equiangular to the triangle DEF; and that the angle ABC will be equal to the *angle* DEF; and the remaining *angle* viz. the *angle* at C equal to the remaining *angle* at F.

For if the angle ABC be unequal to the angle DEF, one of them is greater: let ABC be the greater: and with the straight line AB, and at the point B in it, let the *angle* ABG be made equal to the angle DEF.



And because the angle A is equal to D; and the angle ABG to DEF; therefore the remaining angle AGB is equal to the remaining



Book VI. angle DFE : therefore the triangle ABG is equiangular to the triangle DEF ; therefore it is (by 4. 6.) as AB *is* to BG so *is* DE to EF ; but as DE is to EF so is AB supposed *to be* to BC ; and therefore (by 11. 5.) as AB *is* to BC so *is* AB to BG : wherefore AB has the same ratio to each of the *lines* BC, BG ; therefore (by 9. 5.) BC is equal to BG ; so that also (by 5. 1.) the angle BGC is equal to the angle BCG : but the angle at C is supposed *to be* less than a right angle ; therefore BGC is also less than a right angle ; so that the angle adjacent to it viz. AGB is (by 13. 1.) greater than a right angle ; and it has been demonstrated to be equal to the angle at F ; therefore the angle at F is greater than a right angle ; but it is supposed to be less than a right angle ; which is absurd : wherefore the angle ABC is not unequal to the *angle* DEF, therefore *it is* equal : and also (by supp.) the angle at A is equal to the angle at D ; therefore (by 32. 1.) the remaining *angle* at C is equal to the remaining angle at F : Therefore the triangle ABC is equiangular to the triangle DEF.

But again, let each of the angles at C and F be supposed not *to be* less than a right angle : I say again that also thus the triangle ABC is equiangular to the triangle DEF.

For the same things being constructed, we shall demonstrate in like manner that BC is equal to BG ; so that also (by 5. 1.) the angle at C is equal to the *angle* BGC : but the angle at C is not less than a right angle ; therefore neither is the angle BGC less than a right angle ; therefore the two angles of the triangle BGC are not less than two right angles, which (by 17. 1.) is impossible : Therefore again the angle ABC is not unequal to the *angle* DEF ; therefore *it is* equal : but also (by supp.) the angle at A is equal to the angle at D ; therefore (by 32. 1.) the remaining *angle* at C is equal to the remaining *angle* at F : wherefore the triangle ABC is equiangular to the triangle DEF.

Wherefore if two triangles have one angle equal to one angle ; but the sides about *two* other angles proportionals ; and each of the remaining angles, at the same time, either less or not less than a right angle : the triangles will be equiangular and will have the angles equal about which the sides are, *which are* proportionals. Which was to be demonstrated.

P R O P.



## P R O P. VIII.

Book VI.

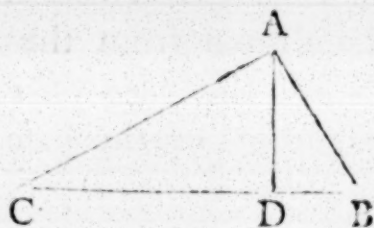
If in a right angled triangle, a perpendicular be drawn from the right angle to the base; the triangles at the perpendicular are similar to the whole *triangle* and to one another.

Let ABC be a right angled triangle, having the angle BAC a right angle; and let AD be drawn from the *point* A perpendicular to BC: I say that each of the triangles ABD, ADC is similar to the whole ABC, and also to one another.

For since the angle BAC is equal to the angle ADB, for each of them is a right angle: and the angle at B is common to the two triangles, both ABC and ABD; therefore (by 32. 1.) the remaining angle ACB is equal to the remaining angle BAD; therefore the triangle ABC is equiangular to the triangle ABD; wherefore it is, (by 4. 6.) as BC subtending the right angle of the triangle ABC is to BA subtending the right angle of the triangle ABD, so is the same AB subtending the angle at C of the triangle ABC to BD subtending the angle equal to that at C, viz. the angle BAD of the triangle ABD: and moreover so is AC to AD subtending the angle at B common to the two triangles: wherefore the triangle ABC is equiangular to the triangle ABD; and has the sides about the equal angles proportionals: therefore (by 1. def. 6.) the triangle ABC is similar to the triangle ABD. Certainly in the same manner we shall demonstrate, that the triangle ADC is also similar to the triangle ABC; therefore each of the triangles ABD, ADC is similar to the whole triangle ABC.

I say also that the triangles ABD, ADC are similar to one another.

For because the right angle BDA is equal to the right angle ADC: but the angle BAD has been demonstrated to be equal to the angle at C; therefore also the remaining angle at B is equal to the remaining angle DAC: wherefore the triangle ABD is equiangular to the triangle ADC: therefore it is, as BD subtending the angle BAD of the triangle ABD is to DA subtending the angle at C, of the triangle ADC, equal to the angle BAD so is the same AD subtending the angle at B of the triangle ABD, to DC subtending





Book VI. tending the angle DAC of the triangle ADC, equal to the *angle* at B : and besides so is BA subtending the right angle ADB to AC subtending the right angle ADC : therefore (by 1. def. 6.) the triangle ABD is similar to the triangle ADC.

Wherefore if in a right angled triangle, a perpendicular be drawn from the right angle to the base ; the triangles at the perpendicular are similar to the whole *triangle* and to one another. Which was to be demonstrated.

Cor. Certainly it is manifest from this, that if a perpendicular be drawn from the right angle in a right angled triangle to the base ; the *perpendicular* drawn is a mean proportional between the segments of the base : and besides the side adjacent to the segment is a mean proportional between the base and any one of the segments.

### P R O P. IX.

To cut off any part required from a given straight line.

Let AB be the given straight line : it is required to cut off any part from AB.

Let a third *part* be required ; and let any straight line AC be drawn from the *point* A, containing any angle, which may accidentally happen, with the line AB ; and let D be taken in AC any point which may accidentally happen ; and (by 3. 1.) let DE, EC be made equal to AD ; and let BC be joined ; and through the point D, let DF be drawn (by 31. 1.) parallel to BC.

Wherefore because DF has been drawn *parallel* to BC one of the sides of the triangle ABC ; therefore (by 2. 6.) there is this proportion, as CD is to DA so is BF to FA : but CD is the double of DA ; therefore BF is the double of FA ; therefore BA is the triple of AF.

Therefore the required third part AF hath been cut off from the given straight line AB. Which was to be done,



P R O P.



## P R O P. X.

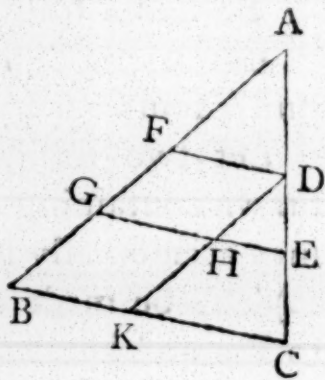
Book VI.

To cut an undivided straight line, in like manner as a given straight line hath been cut.

Let AB be the given undivided straight line, and AC the one which hath been cut : it is required to cut the undivided straight line AB in like manner as AC has been cut.

Let AC be cut in the points D, E : and let *the lines* be placed so as to contain any angle which may accidentally happen ; and let BC be joined ; and let DF, GE be drawn (by 31. 1.) through the points D, E parallel to BC : and through the point D, let DHK be drawn parallel to AB.

Therefore each of the figures FH, HB is a parallelogram : therefore (by 34. 1.) DH is equal to FG ; and HK to GB : And because HE hath been drawn parallel to KC one of the sides of the triangle DKC ; (by 2. 6.) there is *this* proportion, as CE is to ED so is KH to HD ; but KH is equal to BG, and HD to GF ; therefore it is (by 7. 5.) as CE is to ED so is BG to GF : Again, because FD hath been drawn parallel to EG one of the sides of the triangle AGE ; therefore (by 2. 6.) there is *this* proportion, as ED is to DA so is GF to FA ; but it has been also demonstrated that as CE is to ED so is BG to GF : It is therefore as CE is to ED so is BG to GF ; and as ED is to DA so is GF to FA.



Wherefore the given undivided straight line AB hath been cut in like manner as the given straight line AC had been cut. Which was to be done.

## P R O P. XI.

To find a third proportional to two given straight lines.

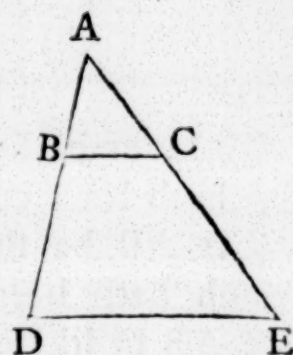
Let AB, AC be the two given straight lines ; and let them be placed containing any angle which may accidentally happen : it is required to find a third proportional to AB, AC.

For let AB, AC be produced to the points D, E ; and let BD be made equal to AC ; and let BC be joined ; and through the point D,



Book VI. D, let DE be drawn parallel to it.

Wherefore because BC hath been drawn *parallel* to DE one of the sides of the triangle ADE, there is *this* proportion (by 2. 6.), as AB is to BD so is AC to CE : but BD is equal to AC (by const.); therefore it is, as AB is to AC so is AC to CE.



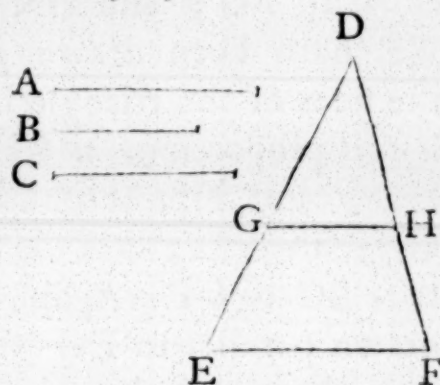
Wherefore two straight lines AB, AC being given, CE a third proportional to them hath been found. Which was to be done.

### P R O P. XII.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines ; it is required to find a fourth proportional to the straight lines A, B, C.

Let two straight lines DE, DF be drawn containing any angle EDF which may accidentally happen : and let (by 3. 1.) DG be made equal to A, GE equal to B ; and besides DH equal to C : and GH being joined, let EF be drawn through E parallel to it.



Therefore because GH hath been drawn parallel to EF one of the sides of the triangle DEF ; therefore it is, (by 2. 6.) as DG is to GE so is DH to HF ; but (by the const.) DG is equal to A, GE to B, and DH to C ; therefore it is, as A is to B so is C to HF.

Wherefore three straight lines A, B, C being given, HF a fourth proportional has been found. Which was to be done.

### P R O P. XIII.

To find a mean proportional between two given straight lines.

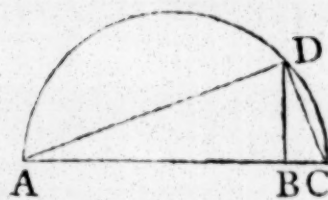
Let AB, BC be the two given straight lines : it is required to find a mean proportional between AB, BC.

Let them be placed in a straight line ; and let the semicircle ADC be described upon AC ; and let BD be drawn from the point B at  
B at



B at right angles to the straight line AC ; and let AD, DC be Book VI.  
joined.

And because the angle ADC is in a semicircle (by 31. 3.) it is a right angle : and because in a right angled triangle ADC, DB hath been drawn from the right angle perpendicular to the base therefore (by cor. to 8. 6.) DB is a mean proportional between AB, BC the segments of the base.



Wherefore two straight lines AB, BC being given, DB a mean proportional *between them* hath been found. Which was to be done.

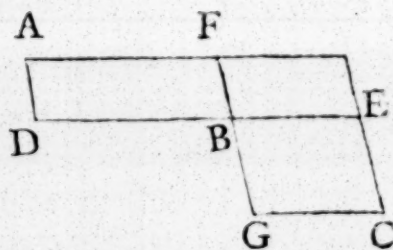
P R O P. XIV.

The sides about the equal angles of equal parallelograms which have one angle equal to one angle are reciprocally proportional : and those parallelograms, having one angle equal to one angle, of which the sides about the equal angles are reciprocally proportional, are equal,

Let AB, BC be equal parallelograms, having the angles at B equal ; and let DB, BE be placed in a straight line ; therefore (by 14. 1.) FB, BG are in a straight line : I say that the sides of the *parallelograms* AB, BC, which are about the equal angles, are reciprocally proportional, that is, as DB is to BE so is GB to BF.

For let the parallelogram FE be compleated.

And because the parallelogram AB is equal (by supp.) to the parallelogram BC ; and FE is some other *parallelogram* ; therefore (by 7. 5.) it is, as the *parallelogram* AB is to the *parallelogram* FE so is the *parallelogram* BC to the *parallelogram* FE :



but as the *parallelogram* AB is to the *parallelogram* FE so (by 1. 6.) is DB to BE : and as the *parallelogram* BC is to the *parallelogram* FE so is GB to BF ; wherefore also (by 11. 5.) as DB is to BE so is GB to BF ; wherefore the sides of the parallelograms AB, BC, *which are* about the equal angles, are reciprocally proportional.

But let the sides about the equal angles be reciprocally proportional, and let it be as DB is to BE so is GB to BF : I say that the parallelogram AB is equal to the parallelogram BC.



Book VI.

For because it is, as DB is to BE so is GB to BF; but as DB is to BE so (by 1. 6.) is the parallelogram AB to the parallelogram FE; and as GB is to BF so is the parallelogram BC to the parallelogram FE; and therefore (by 11. 5.) as the parallelogram AB is to the parallelogram FE so is the parallelogram BC to the parallelogram FE: therefore (by 9. 5.) the parallelogram AB is equal to the parallelogram BC.

Wherefore the sides about the equal angles of equal parallelograms which have one angle equal to one angle, are reciprocally proportional: and those parallelograms, having one angle equal to one angle, of which the sides about the equal angles are reciprocally proportional, are equal. Which was to be demonstrated.

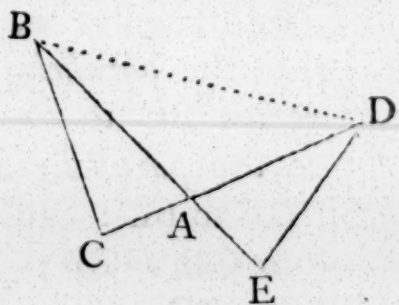
## PROP. XV.

The sides, about the equal angles of equal triangles, which have one angle equal to one angle, are reciprocally proportional: and those triangles having one angle equal to one angle, of which the sides about the equal angles are reciprocally proportional, are equal.

Let ABC, ADE be equal triangles, having one angle equal to one angle; viz. BAC equal to DAE: I say that the sides about the equal angles of the triangles ABC, ADE are reciprocally proportional; that is, as CA is to AD so is EA to AB.

For let the triangles be placed in such a manner, that CA, AD may be in a straight line: therefore also (by 14. 1.) EA, AB are in a straight line: and let BD be joined.

Then because (by supp.) the triangle ABC is equal to the triangle ADE, and ABD is another triangle; therefore it is, as the triangle CAB is to the triangle BAD so is the triangle ADE to the triangle BAD: but (by 1. 6.) as the triangle CAB is to BAD so is CA to AD; and as the



triangle EAD is to the triangle BAD so is EA to AB; therefore also (by 11. 5.) as CA is to AD so is EA to AB: therefore the sides about



about the equal angles of the triangles ABC, ADE are reciprocally proportional. Book VI.

But let the sides of the triangles ABC, ADE be reciprocally proportional; and let it be as CA *is* to AD so *is* EA to AB: I say that the triangle ABC is equal to the triangle ADE.

For again BD being joined: because it is (by supp.) as CA is to AD so *is* EA to AB; but as CA is to AD (by 1. 6.) so *is* the triangle ABC to the triangle BAD; and as EA *is* to AB so *is* the triangle EAD to the triangle BAD: wherefore (by 11. 5.) as the triangle ABC is to the triangle BAD so *is* the triangle EAD to the triangle BAD: therefore each of the triangles ABC, ADE have the same ratio to BAD; therefore (by 9. 5.) the triangle ABC is equal to the triangle EAD.

Wherefore the sides about the equal angles of equal triangles, which have one angle equal to one angle, are reciprocally proportional: and those triangles having one angle equal to one angle, of which the sides about the equal angles are reciprocally proportional, are equal. Which was to be demonstrated.

## P R O P. XVI.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportionals.

Let the four straight lines AB, CD, E, F be proportionals, viz. as AB *is* to CD so *let* E *be* to F: I say that the rectangle contained by AB and F is equal to the rectangle contained by CD and E.

For from the points A, C let AG, CH be drawn at right angles to the straight lines AB, CD; and let AG be made equal to F; and CH equal to E; and let the parallelograms BG, DH be completed.

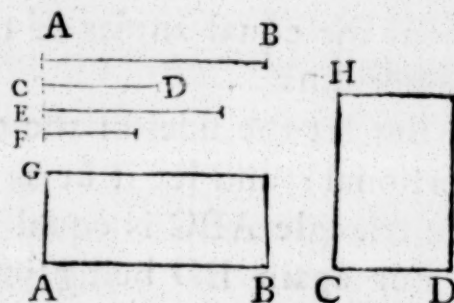
And because it is, as AB *is* to CD so *is* E to F; and CH *is* equal to E and AG to F: therefore it is (by 7. 5.) as AB *is* to CD so *is* CH to AG; therefore the sides about the equal angles of the parallelograms BG, DH are reciprocally proportional; but those equi-



Book VI. angular parallelograms are equal (by

14. 6.) of which the sides about the equal angles are reciprocally proportional: therefore the parallelogram BG is equal to the parallelogram DH: and the *parallelogram* BG is the *rectangle* contained by AB and F; for AG is

equal to F: and the *parallelogram* DH is the rectangle contained by CD and E; for CH is equal to E: wherefore the rectangle contained by AB and F is equal to the rectangle contained by CD and E.



But let the rectangle contained by AB and F be equal to the rectangle contained by CD and E: I say that the four straight lines will be proportionals, viz. as AB is to CD so will E be to F.

For the same things being constructed; because (by the supp.) the rectangle contained by AB, F is equal to the rectangle contained by CD, E; and the *parallelogram* BG is the rectangle contained by AB, F; for AG is equal to F: and the *parallelogram* DH is the rectangle contained by CD, E; for CH is equal to E: therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular: but (by 14. 6.) the sides about the equal angles of equal and equiangular parallelograms are reciprocally proportional: wherefore it is, as AB is to CD so is CH to AG: but CH is equal to E; and AG to F: therefore it is as AB is to CD so is E to F.

Wherefore if four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportionals. Which was to be demonstrated.

#### PROP. XVII.

If three straight lines be proportionals; the rectangle contained by the extremes is equal to the square of the mean *proportional*: and if the rectangle contained by the extremes be equal to the square of the mean *term*, the three straight lines will be proportionals.

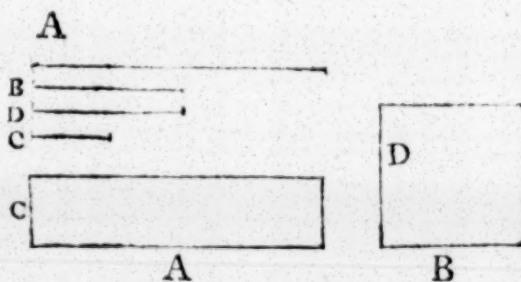
Let



Let the three straight lines A, B, C be proportionals; viz. as A <sup>Book VI.</sup> is to B so let B be to C: I say that the rectangle contained by A and C is equal to the square of the mean B.

Let D be made equal to B.

And because it is, as A is to B so is B to C; and B is equal to D; therefore it is, as A is to B so is D to C: but if four straight lines be proportionals, the rectangle contained by the extremes is equal (by 16. 6.) to the rectangle contained



by the means; therefore the rectangle contained by A and C is equal to the rectangle contained by B and D: but the rectangle contained by B and D is equal to the square of B; for B is equal to D: therefore the rectangle contained by A and C is equal to the square of B.

But let the rectangle contained by A and C be equal to the square of B; I say that it is, as A is to B so is B to C.

For the same things being constructed; because (by supp.) the rectangle contained by A and C is equal to the square of B; but the square of B is equal to the rectangle contained by B and D; for B is equal to D: therefore the rectangle contained by A and C is equal to the rectangle contained by B and D; but (by 16. 6.) if the rectangle contained by the extremes be equal to the rectangle contained by the means; the four straight lines are proportionals: therefore it is, as A is to B so is D to C; but B is equal to D; therefore as A is to B so is B to C.

Wherefore if three straight lines be proportionals; the rectangle contained by the extremes is equal to the square of the mean *proportional*: and if the rectangle contained by the extremes be equal to the square of the mean *term*; the three straight lines will be proportionals. Which was to be demonstrated.

PROP. XVIII.

Upon a given straight line to describe a rectilineal figure, similar and similarly situated to a given rectilineal figure.

\* F 2

Let



Book VI. Let  $AB$  be the given straight line; and  $CE$  the given rectilinear figure: it is required, upon the given straight line  $AB$  to describe a rectilinear figure similar and similarly situated to the rectilinear figure  $CE$ .

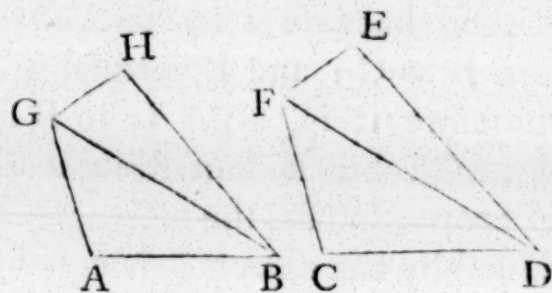
Let  $DF$  be joined; and with the straight line  $AB$  and at the points  $A, B$  in it, let the angle  $GAB$  be made equal to the angle at  $C$ ; and the angle  $ABG$  equal to the angle  $CDF$ ; therefore (by the 32. 1.) the remaining angle  $CFD$  is equal to the remaining angle  $AGB$ ; therefore the triangle  $FCD$  is equiangular to the triangle  $GAB$ : therefore there is this proportion (by 4. 6.) as  $FD$  is to  $GB$  so is  $FC$  to  $GA$ ; and so is  $CD$  to  $AB$ : Again, with the straight line  $BG$ , and at the points  $B, G$  in it, let the angle  $BGH$  be made equal to the angle  $DFE$ ; and the angle  $GBH$  equal to the angle  $FDE$ ; therefore the remaining angle at  $H$  is equal to the remaining angle at  $E$ : therefore the triangle  $FDE$  is equiangular to the triangle  $GBH$ ; therefore there is this proportion (by 4. 6.) as  $FD$  is to  $GB$  so is  $FE$  to  $GH$ ; and so is  $ED$  to  $HB$ : But it has been also demonstrated that as  $FD$  is to  $GB$  so is  $FC$  to  $GA$ , and so is  $CD$  to  $AB$ : therefore (by 11. 5.) as  $FC$  is to  $GA$  so is  $CD$  to  $AB$  and so is  $FE$  to  $GH$ , and besides so is  $ED$  to  $HB$ : And because the angle  $CFD$  is equal to the angle  $AGB$  and  $DFE$  is equal to  $BGH$ ; therefore the whole angle  $CFE$  is equal to the whole  $AGH$ . Certainly for the same reason also the angle  $CDE$  is equal to  $ABH$ : but also the angle at  $C$  is equal to the angle at  $A$ ; and the angle at  $E$  to the angle at  $H$ : therefore the figure  $AH$  is equiangular to the figure  $CE$ ; and has the sides about the equal angles in it proportionals; therefore (by 1. def. 6.) the rectilinear figure  $AH$  is similar to the rectilinear figure  $CE$ .

Wherefore upon a given straight line  $AB$  a rectilinear figure  $AH$  hath been described, similar and similarly situated to the given rectilinear figure  $CE$ . Which was to be done.

#### P R O P. XIX.

Similar triangles are to one another in the duplicate ratio of the sides of like ratio.

Let

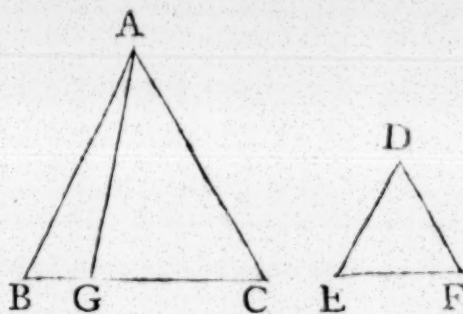




Let  $ABC, DEF$  be similar triangles, having the angle at  $B$  equal <sup>Book VI.</sup> to the angle at  $E$ : and as  $AB$  is to  $BC$  so let  $DE$  be to  $EF$ ; so that (by 12. def. 5.)  $BC$  may be the side of like ratio to  $EF$ : I say that the triangle  $ABC$  has to the triangle  $DEF$  the duplicate ratio of that which  $BC$  has to  $EF$ .

For (by 11. 6.) let  $BG$  be taken a third proportional to  $BC, EF$ ; so that it may be, as  $BC$  is to  $EF$  so is  $EF$  to  $BG$ : and let  $GA$  be joined,

Wherefore because it is, as  $AB$  is to  $BC$  so is  $DE$  to  $EF$ ; therefore alternately (by 16. 5.) it is, as  $AB$  is to  $DE$  so is  $BC$  to  $EF$ : but (by const.) as  $BC$  is to  $EF$  so is  $EF$  to  $BG$ : therefore also (by 11. 5.) as  $AB$  is to  $DE$  so is  $EF$  to  $BG$ : therefore the sides about the equal angles of the triangles



$ABG, DEF$  are reciprocally proportional: but those *triangles* having one angle equal to one angle, and of which the sides about the equal angles are reciprocally proportional, are equal. Therefore the triangle  $ABG$  is equal to the triangle  $DEF$ : and because it is, as  $BC$  is to  $EF$  so is  $EF$  to  $BG$ ; and if three straight lines be proportionals (by 10. def. 5.) the first is said to have to the third a duplicate ratio of that which it *has* to the second: therefore  $BC$  has to  $BG$  a duplicate ratio of that which  $BC$  *has* to  $EF$ : but as  $BC$  is to  $BG$  so (by 1. 6.) is the triangle  $ABC$  to the triangle  $ABG$ : therefore also the triangle  $ABC$  has to the triangle  $ABG$  the duplicate ratio of that which  $BC$  *has* to  $EF$ : but the triangle  $ABG$  is equal to the triangle  $DEF$ ; therefore (by 7. 5.) the triangle  $ABC$  has to the triangle  $DEF$  the duplicate ratio of that which  $BC$  *has* to  $EF$ .

Wherefore similar triangles are to one another in the duplicate ratio of the sides of like ratio. Which was to be demonstrated.

Cor. Certainly from this it is manifest, that if three straight lines be proportionals, it is, as the first is to the third so is a triangle described upon the first, to a similar and similarly situated *triangle* described upon the second; since it has been demonstrated, as  $CB$  is to  $BG$  so is the triangle  $ABC$  to the triangle  $ABG$ , that is  $DEF$ .

P R O P.



Similar polygons are divided into similar triangles, and into the same number, and *have the same ratio to one another which the whole polygons have* : and the *one* polygon has to the *other* polygon the duplicate ratio of that which one side of the one has to the side of like ratio of the other.

Let  $ABCDE$ ,  $FGHKL$  be similar polygons, and let  $AB$  be the side of like ratio  $FG$  : I say that the polygons  $ABCDE$ ,  $FGHKL$  are divided into similar triangles ; and into the same number ; and have the same ratio *to one another which the whole polygons have* ; and *that* the polygon  $ABCDE$  has to the polygon  $FGHKL$  the duplicate ratio of that which  $AB$  has to  $FG$ .

Let  $BE$ ,  $EC$  ;  $GL$ ,  $LH$  be joined.

And because the polygon  $ABCDE$  is similar to the polygon  $FGHKL$ , the angle  $BAE$  is equal to the *angle*  $GFL$  : and it is, as  $BA$  is to  $AE$  so is  $GF$  to  $FL$  : wherefore because there are two triangles  $ABE$ ,  $FGL$  having one angle equal to one angle, and the sides about the equal angles proportionals ; therefore (by 6. 6.) the triangle  $ABE$  is equiangular to the triangle  $FGL$  ; so that (by 4. 6.) it is also similar : therefore the angle  $ABE$  is equal to the angle  $FGL$  : and also on account of the similarity of the polygons, the whole *angle*  $ABC$  is equal to the whole *angle*  $FGH$  ; therefore the remaining angle  $EBC$  is equal to the remaining *angle*  $LGH$  ; and because, on account of the similarity of the triangles  $ABE$ ,  $FGL$ , it is, as  $EB$  is to  $BA$  so is  $LG$  to  $GF$  ; but also, on account of the similarity of the polygons, it is, as  $AB$  is to  $BC$  so is  $FG$  to  $GH$  : therefore by equality it is, (by 22, 5.) as  $EB$  is to  $BC$  so is  $LG$  to  $GH$  ; and the sides *therefore* about the equal angles  $EBC$ ,  $LGH$  are proportionals ; therefore the triangle  $EBC$  is equiangular to the triangle  $LGH$  (by 6. 6.) ; so that *it is* also similar (by 4. 6.) : Certainly for the same reason the triangle  $ECD$  is similar to the triangle  $LHK$  : therefore the similar polygons  $ABCDE$ ,  $FGHKL$  are divided into similar triangles, and into an equal number.

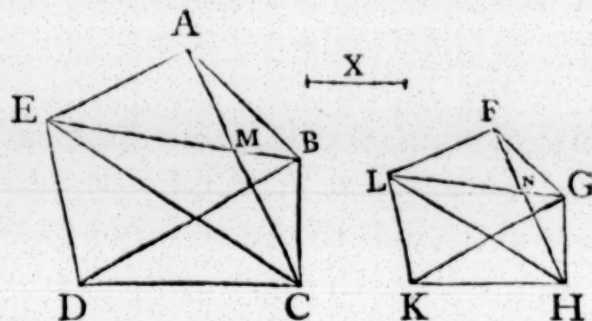
I say also that they have the same ratio to one another which the whole polygons have : that is, so that the triangles are proportionals ; and that the antecedents are the *triangles*  $ABE$ ,  $EBC$ ,  $ECD$



ECD and the consequents of them, the *triangles* FGL, LGH, <sup>Book VI.</sup> LHK : also that the polygon ABCDE has to the polygon FGHLK the duplicate ratio of that which the side of like ratio of *the one* has to the side of like ratio of *the other*, that is, which AB has to FG.

For let AC, FH be joined.

And because, on account of the similarity of the polygons, the angle ABC is equal to the *angle* FGH, and it is as AB is to BC so is FG to GH ; the triangle ABC is equiangular (by 6. 6.) to the



triangle FGH : therefore the *angle* BAC is equal to the *angle* GFH ; and BCA to GHF : and because the angle BAM is equal to GFN ; and the angle ABM has been demonstrated to be equal to FGN ; therefore the remaining *angle* AMB is equal (by 32. 1.) to the remaining *angle* FNG ; therefore the triangle ABM is equiangular to the triangle FGN : Certainly in the same manner we shall demonstrate that the *triangle* BMC is equiangular to the triangle GNH ; therefore there is this proportion, (by 4. 6.) as AM is to MB so is FN to NG ; and as BM is to MC so is GN to NH : so that also by equality (by 22. 5.) as AM is to MC so is FN to NH : but as AM is to MC so (by 1. 6.) is the triangle ABM to the *triangle* BMC ; and so is the triangle AEM to the *triangle* MEC ; for they are to one another as *their* bases : And (by 12. 5.) as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents ; therefore as the triangle AMB is to the triangle BMC so is the *triangle* ABE to the *triangle* CBE : but as the *triangle* AMB is to the *triangle* BMC so is AM to MC : therefore also (by 11. 5.) as AM is to MC so is the triangle ABE to the triangle CBE. Certainly for the same reason also as FN is to NH so is the triangle FGL to the triangle GLH : and it is, as AM is to MC so is FN to NH ; therefore (by 11. 5.) as the triangle ABE is to the triangle CBE so is the triangle FGL to the triangle GLH ; and alternately (by 16. 5.) as the triangle ABE is to the triangle FGL so is the triangle CBE to the triangle GLH.

Cer-



Book VI. Certainly in the same manner we shall demonstrate, BD and GK being joined, that also, as the triangle BEC is to the triangle GLH so is the triangle ECD to the triangle LHK : and because it is, as the triangle ABE is to the triangle FGL so is the triangle EBC to the triangle LGH, and besides so is the triangle ECD to the triangle LHK : wherefore also (by 12. 5.) as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents ; therefore as the triangle ABE is to the triangle FGL so is the polygon ABCDE to the polygon FGHLK : but the triangle ABE has to the triangle FGL (by 19. 6.) the duplicate ratio of that which the side of like ratio AB has to the side of like ratio FG ; for similar triangles are in the duplicate ratio of the sides of like ratio ; therefore also the polygon ABCDE has to the polygon FGHLK the duplicate ratio of that which the side of like ratio AB has to the side of like ratio FG.

Wherefore similar polygons are divided into similar triangles, and into an equal number of them, and of like ratio with the whole polygons ; and the *one* polygon has to the *other* polygon the duplicate ratio of that which *one* side of like ratio has to another side of like ratio. Which was to be demonstrated.

Cor. 1. Certainly in like manner it will be demonstrated in quadrilateral figures, that they are in the duplicate ratio of the sides of like ratio ; and it has been demonstrated also in triangles ; so that universally similar rectilineal figures are to one another in the duplicate ratio of the sides of like ratio.

Cor. 2. And if we take the *line* X (by 11. 6.) a third proportional to AB, FG ; (by 10. def. 5.) AB has to X the duplicate ratio of that which AB has to FG : and also a polygon to a polygon, and a quadrilateral *figure* to a quadrilateral *figure* has the duplicate ratio of that which the side of like ratio has to the side of like ratio, that is, *which* AB has to FG : and this has been demonstrated also in triangles ; so that it is also manifest universally, that if three straight lines be proportionals, it will be, as the first is to the third so is a *rectilineal* figure described upon the first to a similar and similarly situated *one* described upon the second.

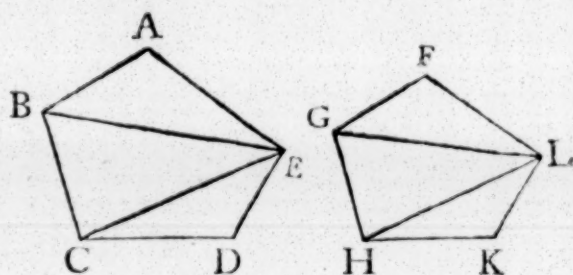
OTHERWISE. We shall demonstrate in a different manner and more expeditiously, that the triangles are of like ratio.

For



For again let the polygons ABCDE, FGHL be supposed to <sup>Book VI.</sup> be described; and let BE, EC; GL, LH be joined: I say that it is, as the triangle ABE is to the triangle FGL so is the triangle EBC to the triangle LGH, and CDE to HKL.

For because the triangle ABE is similar to the triangle FGL; therefore (by 19. 6.) the triangle ABE has to the triangle FGL the duplicate ratio of that which BE has to GL. Certainly for the same reason also the triangle BEC has to the triangle GLH the duplicate ratio of that which BE has to GL: therefore it is (by 11. 5.) as the triangle ABE is to the triangle FGL so is the triangle EBC to the triangle LGH. Again, because the triangle EBC is similar to the triangle LGH; therefore EBC has to LGH the duplicate ratio of that which the straight line CE has to the straight line HL. Certainly for the same reason also, the triangle ECD has to the triangle LHK the duplicate ratio of that which CE has to HL; therefore it is, (by 11. 5.) as the triangle EBC is to the triangle LGH so is the triangle ECD to the triangle LHK: but it has been demonstrated that as the triangle EBC is to LGH so is ABE to FGL; therefore also as ABE is to FGL so is BEC to LGH, and so is ECD to LHK: wherefore also (by 12. 5.) as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents; and the remainder as in the former demonstration. Which was to be demonstrated.



PROP. XXI.

Figures which are similar to the same rectilineal figure; are also similar to one another.

For let each of the rectilineal figures A, B be similar to C: I say that A is also similar to B.

For because A is similar to C, it is also equiangular to it, and has the sides about the equal angles proportionals (by 1. def. 6.). Again be-





Book VI. *cause B is similar to C ; it is also equiangular to it, and has the sides about the equal angles proportionals : therefore each of the figures A, B are equiangular to C and have the sides about their equal angles proportionals to the sides about the equal angles of the figure C ; wherefore the figure A is equiangular to B (by com. not. 1.) and has (by 11. 5.) the sides about the equal angles proportionals : wherefore (by 1. def. 6.) the figure A is similar to the figure B. Which was to be demonstrated.*

## PROP. XXII.

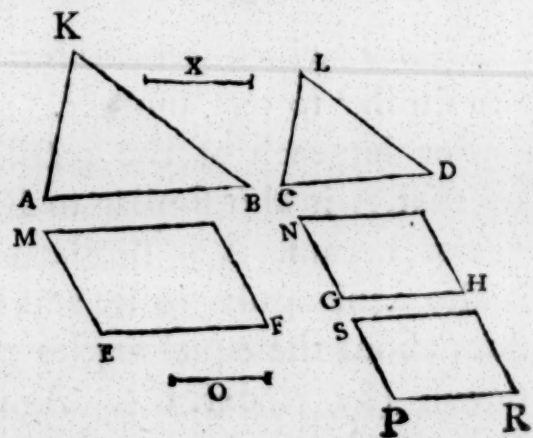
If four straight lines be proportionals ; also the rectilineal figures similar and similarly described upon them will be proportionals : and if the rectilineal figures similar and similarly described upon them be proportionals ; also the straight lines themselves will be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals, viz. as AB is to CD so let EF be to GH ; and let the rectilineal figures KAB, LCD be described upon AB, CD similar and similarly situated ; and the rectilineal figures MF, NH upon EF, GH similar and similarly situated : I say that it is, as KAB is to LCD so is MF to NH.

For let X be taken (by 11. 6.) a third proportional to AB, CD : and O a third proportional to EF, GH : and because it is, as AB is to CD so is EF to GH ; and as CD is to X so is GH to O ; therefore by equality it is, (by 22. 5.) as AB is to X so is EF to O : But as AB is to X so (by cor. 2. to 20. 6.) is the figure KAB to LCD ; and as EF is to O so is the figure MF to NH ; therefore (by 11. 5.) as the figure KAB is to LCD so is the figure MF to NH.

But let it be, as the figure KAB is to LCD so let the figure MF be to NH : I say that it is, as AB is to CD so is EF to GH.

For as AB is to CD so let EF be made (by 12. 6.) to PR ; and let the rectilineal figure SR be described (by 18. 6.) upon PR similar and similarly situated to either of the figures MF, NH.





Wherefore because it is, as AB *is* to CD so *is* EF to PR ; and Book VI.  
the figures KAB, LCD have been described upon AB *and* CD similar and similarly situated ; and the figures MF *and* SR similar and similarly situated upon EF *and* PR ; therefore it is (by part. 1. of this.) as the figure KAB *is* to LCD so *is* the figure MF to SR ; but it is also supposed as the figure KAB *is* to LCD so *is* the figure MF to NH ; therefore (by 11. 5.) the figure MF has the same ratio to each of the figures NH and SR ; therefore (by 9. 5.) NH is equal to SR ; but it is similar to it and similarly situated : therefore (by the following lemma) GH is equal to PR ; and because it is, as AB *is* to CD so *is* EF to PR (by const.) : and PR is equal to GH ; therefore it is, as AB *is* to CD so *is* (by 7. 5.) EF to GH.

Wherefore if four straight lines be proportionals ; also the similar rectilineal figures similarly described upon them will be proportionals : and if the similar figures similarly described upon them be proportionals ; also the straight lines themselves will be proportionals. Which was to be demonstrated.

LEMMA. If rectilineal figures be equal and similar, we shall thus demonstrate, that the sides of them, which are of like ratio, are equal.

Let NH and SR be equal and similar rectilineal figures ; and let it be, as HG *is* to GN so *is* RP to PS : I say that PR is equal to HG.

For if they are unequal, one of them is greater ; let RP be greater than HG : and because it is, as RP *is* to PS so *is* HG to GN, and alternately, as RP *is* to HG so *is* PS to GN : but PR is greater than GH ; therefore also PS is greater than GN : so that also (by 20. 6.) the figure RS is greater than the figure HN ; but it is also equal, which is impossible ; therefore PR is not unequal to GH ; therefore equal. Which was to be demonstrated.

P R O P. XXIII.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG : I say that the parallelogram AC

\* G 2

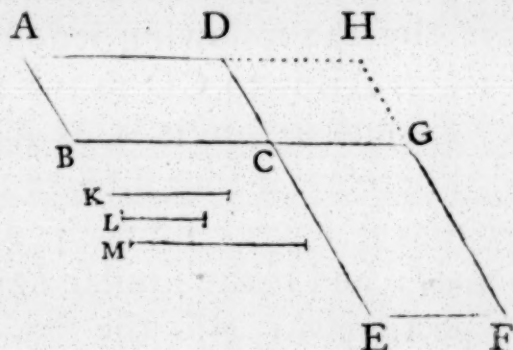
has



Book VI. has to the parallelogram CF the ratio compounded of the ratios of the sides; of that which BC has to CG; and of that which DC has to CE.

For let them be placed so that BC may be in a straight line with CG; therefore (by 14. 1.) DC is in a straight line with CE: and let the parallelogram DG be completed; and let K any straight line be drawn; and, as BC is to CG so (by 12. 6.) let K be made to L; and as DC is to CE so let L be made to M.

Therefore (by const.) the ratios of K to L and of L to M, are the same ratios with the ratios of the sides; viz. of BC to CG and of DC to CE: but the ratio of K to M is compounded both of the ratio of K to L and of the ratio of L to M; so that also K has to M the ratio compounded of the ratios of the sides.



And because it is (by 1. 6.) as BC is to CG so is the parallelogram AC to the parallelogram CH: but as BC is to CG so (by const.) is K to L; therefore also (by 11. 5.) as K is to L so is the parallelogram AC to the parallelogram CH: Again because it is, (by 1. 6.) as DC is to CE so is the parallelogram CH to the parallelogram CF: but as DC is to CE so (by const.) is L to M; therefore also (by 11. 5.) as L is to M so is the parallelogram CH to the parallelogram CF: wherefore because it has been demonstrated, that as K is to L so is the parallelogram AC to the parallelogram CH; and as L is to M so is the parallelogram CH to the parallelogram CF; therefore by equality it is, (by 22. 5.) as K is to M so is the parallelogram AC to the parallelogram CF; and K has to M the ratio compounded of the ratios of the sides; therefore also, the parallelogram AC has to the parallelogram CF the ratio compounded of the ratios of the sides.

Wherefore equiangular parallelograms have to one another the ratio compounded of the ratios of their sides. Which was to be demonstrated.

PROP.



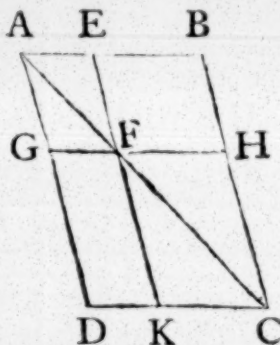
## P R O P. XXIV.

Book VI.

The parallelograms, which are about the diameter of every parallelogram, are similar to the whole and to one another.

Let ABCD be a parallelogram ; and AC its diameter ; and let EG, HK be parallelograms about the *diameter* AC : I say that each of the parallelograms EG, HK is similar to the whole *parallelogram* ABCD and to one another.

For because EF has been drawn parallel to BC one of the sides of the triangle ABC, there is this proportion (by 2. 6.), as BE is to EA so is CF to FA : again because FG hath been drawn parallel to CD one of the sides of the triangle ADC ; therefore there is this proportion, as CF is to FA so is DG to GA : but as CF is to FA so has BE been demonstrated to be to EA : therefore also (by 11. 5.) as BE is to EA so is DG to GA ; and by composition therefore (by 18. 5.) as BA is to AE so is DA to AG ; and alternately (by 16. 5.) as BA is to AD so is EA to AG : therefore the sides about the common angle BAD of the parallelograms ABCD and EG are proportionals. And because GF is parallel to DC, the angle AGF (by 29. 1.) is equal to ADC ; and the angle GFA to DCA ; and the angle DAC is common to the two triangles ADC, AGF ; therefore the triangle ADC is equiangular to the triangle AGF. Certainly for the same reason also the triangle ABC is equiangular to the triangle AFE : therefore also the whole parallelogram ABCD is equiangular to the parallelogram EG : therefore there is this proportion, (by 4. 6.) as AD is to DC so is AG to GF ; and as DC is to CA so is GF to FA ; and as AC is to CB so is AF to FE and besides as CB is to BA so is FE to EA : and because it has been demonstrated that as DC is to CA so is GF to FA ; and as AC is to CB so is AF to FE ; therefore by equality it is, (by 22. 5.) as DC is to CB so is GF to FE : therefore the sides about the equal angles of the parallelograms ABCD and EG are proportionals : therefore (by 1. def. 6.) the parallelogram ABCD is similar to the parallelogram EG. Certainly for the same reason also the parallelogram ABCD is similar to the parallelogram HK ;  
there-





Book VI. therefore each of the parallelograms EG, HK is similar to the parallelogram ABCD : but (by 21. 6.) *figures* which are similar to the same rectilinear *figure* are also similar to one another : therefore also the parallelogram EG is similar to the parallelogram HK.

Wherefore the parallelograms which are about the diameter of every parallelogram are similar to the whole, and to one another. Which was to be demonstrated.

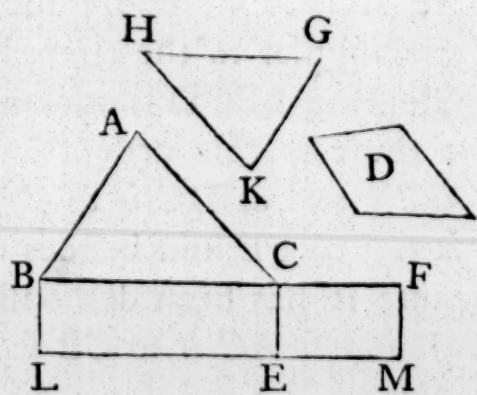
P R O P. XXV.

To make the same *rectilinear figure* similar to a given rectilinear figure, and equal to another given *rectilinear figure*.

Let ABC be the given rectilinear figure, to which it is required that the *figure* should be made similar ; and D the *one* to which it is to be equal : it is required that the same figure be made similar to ABC and equal to D.

For let the parallelogram BE be applied to the straight line BC (by 44. 1.) equal to the triangle ABC ; and (by 45. 1.) let the parallelogram CM be applied to the straight line CE, equal to the *figure* D, in the angle FCE which is equal to the angle CBL ; therefore BC is (by 14. 1.) in a straight line with CF ; and LE with EM ; and let GH be taken (by 13. 6.) a mean proportional between BC, CF : and let the *figure* KGH be described (by 18. 6.) upon GH similar to ABC, and similarly situated.

And because it is, as BC is to GH so is GH to CF (by const.) ; and (by 2. Cor. to 20. 6.) if three straight lines be proportionals, it is, as the first is to the third so is the *rectilinear figure* upon the first to a similar and similarly described *figure* upon the second ; therefore it is, as BC is to CF so is the triangle ABC to KGH ; but also (by 1. 6.) as BC is to CF so is the parallelogram BE to the parallelogram EF ; therefore also (by 11. 5.) as the triangle ABC is to the triangle KGH so is the parallelogram BE to the parallelogram EF : alternately (by 16. 5.) there-





therefore as the triangle ABC is to the parallelogram BE so is the triangle KGH to the parallelogram EF : but (by const.) the triangle ABC is equal to the parallelogram BE ; therefore also the triangle KGH is equal to the parallelogram EF : but the parallelogram EF is equal to the rectilineal figure D : therefore also KGH is equal to D ; and KGH is also similar to ABC.

Wherefore, the same figure KGH has been made similar to the given rectilineal figure ABC and equal to another given figure. Which was to be done.

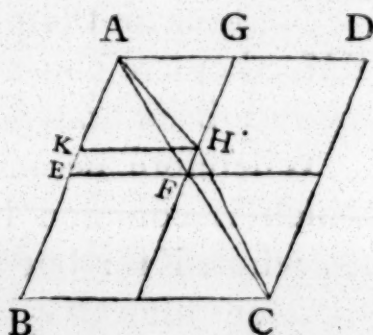
P R O P. XXVI.

If a parallelogram be taken away from a parallelogram similar to the whole and similarly situated, having a common angle with it ; it is about the same diameter with the whole parallelogram.

For let the parallelogram AEFG be taken away from the parallelogram ABCD, similar to ABCD, and similarly situated, having the angle DAB common with it : I say that ABCD is about the same diameter with AEFG.

For if not, but if possible, let AHC be the diameter [of them] of the parallelogram BD ; and let HK be drawn through the point H parallel to either of the lines AD, BC.

Wherefore because ABCD is about the same diameter with the parallelogram KG ; ABCD is (by 24. 6.) similar to the parallelogram KG ; therefore it is (by 1. def. 6.) as DA is to AB so is GA to AK ; but it is also, on account of the similarity of the parallelograms ABCD, EG as DA is to AB so is GA to AE ; therefore also (by 11. 5.) as



GA is to AE so is GA to AK : therefore GA has the same ratio to each of the lines AE, AK ; therefore (by 9. 5.) AE is equal to AK ; the less to the greater ; which is impossible : therefore ABCD is not about the same diameter with the parallelogram KG : therefore the parallelogram ABCD is about the same diameter with the parallelogram AEFG.

Wherefore if a parallelogram be taken away from a parallelogram similar to the whole and similarly situated, having a common angle with

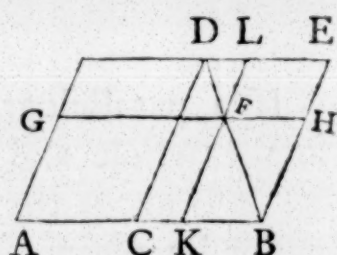


Book VI. with it; it is about the same diameter with the whole *parallelogram*. Which was to be demonstrated.

## P R O P. XXVII.

Of all the parallelograms applied to the same straight line, and deficient by parallelogram figures similar and similarly situated to the *parallelogram* applied to half the line: the parallelogram applied to half the line, *as* being similar to the defect, is the greatest.

Let AB be a straight line; and let it be cut in halves in the point C: and let the parallelogram AD be applied to the straight line AB; deficient by the parallelogram figure CE, similar and similarly situated to that described upon the half of the *straight line* AB, that is upon BC: I say that AD is the greatest of all the parallelograms applied to the straight line AB, and deficient by parallelogram figures similar and similarly situated to CE. For let the parallelogram AF be applied to the straight line AB, deficient by the parallelogram figure KH, similar and similarly situated to CE; I say that AD is greater than AF.



For because the parallelogram CE is similar to the parallelogram KH, they are (by 26. 6.) about the same diameter: let their diameter DB be drawn: and let the figure be described.

Wherefore because CF is equal (by 43. 1.) to FE, let KH which is common be added, therefore the whole CH is equal to the whole KE: but CH is equal to CG, because AC is equal to CB; therefore also CG is equal to EK: let CF which is common be added; therefore the whole AF is equal to the gnomon CHL: so that the parallelogram CE, that is AD (by 36. 1.) is greater than the parallelogram AF.

But again let the straight line AB be cut in halves in the point C; and AL being applied deficient by the figure CM; and again let the parallelogram AE be applied to AB deficient by DF similar and similarly situated to CM the *parallelogram* applied to half the line AB: I say that AL the *parallelogram* applied to half the line is greater than AE.

For



For because DF is similar to GM, they are (by 26. 6.) about the same diameter; let their diameter be EB; and let the figure be described.

And because LF is equal to LH (by 36. 1.) because FG is equal to GH; therefore LF is equal to DL; therefore DL is greater than EK; let DK *which is common* be added; therefore the whole AL is greater than the whole AE.



Wherefore of all the parallelograms applied to the same straight line, and deficient by parallelogram figures similar and similarly situated to the *parallelogram* applied to half the line: the parallelogram applied to half the line, *as* being similar to the defect, is the greatest. Which was to be demonstrated.

P R O P. XXVIII.

To apply to a given straight line a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram figure, which is similar to a given parallelogram: but it is required, that the given rectilineal figure to which it is required that the one to be applied is to be equal, should not be greater than the parallelogram applied to half the line; these being similar, the deficient *parallelogram*, the one applied to half the line and the parallelogram to which the deficient one is required to be similar.

Let AB be the given straight line; and C the given rectilineal figure, to which it is required to apply to AB one equal; not being greater than the parallelogram applied to half the line, the defects being similar; and let D be the *parallelogram* to which the defect is required to be similar: it is required to apply to the given straight line AB a parallelogram equal to the given rectilineal figure C; deficient by a parallelogram figure which is similar to D.

Let AB be cut in halves at the point E; and let ECFG be described upon EB similar to D, and similarly situated: and let the parallelogram AG be completed: certainly AG is either equal to C, or greater than it, on account of the limitation: if therefore AG be equal to C, what was required hath been done; for the parallelogram AG hath been applied to the given straight line AB, equal to the given rectilineal figure C, deficient by the paral-

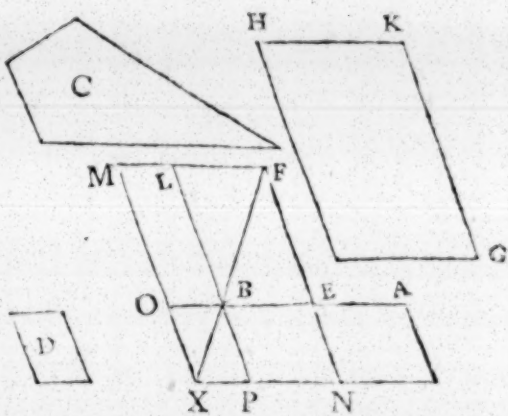






Let AB be the given straight line ; and C the given rectilineal figure to which the *parallelogram* applied to AB is required to be equal : and let D be the *parallelogram* to which it is required that the exceeding one be similar : it is required to apply a parallelogram to the straight line AB equal to the rectilineal figure C exceeding by a parallelogram figure similar to D. Book VI.

Let AB be cut in halves in the point E : and let the parallelogram EL be described upon EB similar to D and similarly situated ; and let GH be made equal to both, the *parallelogram* EL and the *figure* C together, and also similar to D, and the same similarly situated (by 25. 6.) : therefore GH is similar to EL : but let KH be the *side* of like ratio with FL ; and KG the



*side* of like ratio with FE : and because the *parallelogram* GH is greater than EL ; therefore the *side* KH is greater than FL ; and KG than FE : Let FL, FE be produced ; and let FLM be equal to KH, and FEN equal to GK ; and let the *parallelogram* MN be compleated ; therefore MN is equal and similar to GH : but GH is also similar to EL ; therefore also (by 21. 6.) MN is similar to EL : therefore (by 26. 6.) EL is about the same diameter with MN : let their diameter FX be drawn ; and let the figure be described.

Wherefore because the *parallelogram* GH is equal to EL and C together (by const.) ; but GH is equal to MN, therefore also MN is equal to EL and C together : let EL which is common be taken away ; therefore the remaining gnomon NOL is equal to the figure C : and because EA is equal to EB ; the *parallelogram* AN is also (by 36. 1.) equal to NB, that is (by 43. 1.) to LO ; let EX which is common be added ; therefore the whole AX is equal to the gnomon NOL ; but the gnomon NOL is equal to the *figure* C ; therefore AX is also equal to the *figure* C.

Wherefore the parallelogram AX has been applied to the given straight line AB, equal to the given rectilineal figure C ; exceeding by the parallelogram figure PO which is similar to D ; because (by 24. 6.) EL is similar to OP. Which was to be done.

\* H 2

P R O P.

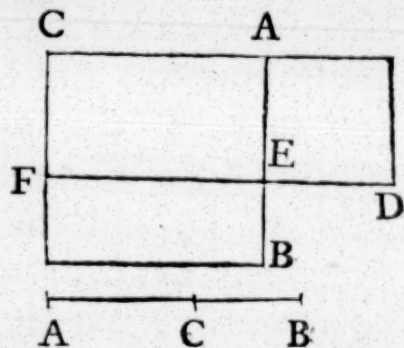


To cut a given finite straight line in extreme and mean ratio.

Let AB be the given finite straight line: it is required to cut the straight line AB in extreme and mean ratio.

For let the square BC be described upon AB; and (by 29. 6.) let the parallelogram CD be applied to AC, equal to the square BC, exceeding by the figure AD similar to BC.

But BC is a square, therefore AD is also a square: and because BC is equal to CD, let CE *which is* common be taken away; therefore the remainder BF is equal to the remainder AD; and it is also equiangular to it; therefore (by 14. 6.) the sides which are about the equal angles of the figures BF, AD are reciprocally proportional; therefore it is, as FE is to ED so is AE to EB; but (by 34. 1.) FE is equal to AC, that is to AB, and ED to AE: therefore it is, as AB is to AE so is AE to EB: But AB is greater than AE; therefore AE is greater than EB.



Wherefore (by 3. def. 6.) the straight line AB is cut in extreme and mean ratio at the point E; and AE is the greater segment of it. Which was to be done.

OTHERWISE. Let AB be the given straight line; it required to cut the straight line AB in extreme and mean ratio.

For let AB be cut in C (by 11. 2.) so that the rectangle contained by AB, BC may be equal to the square of AC.

Wherefore because the rectangle contained by AB, BC is equal to the square of AC; therefore it is, (by 17. 6.) as AB is to AC so is AC to CB; therefore AB hath been cut in extreme and mean ratio in the point C. Which was to be done.

In right angled triangles, the figure described upon the side subtending the right angle is equal to the figures similar and similarly described upon the sides containing the right angle.

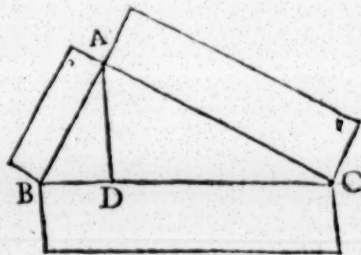
Let



Let  $ABC$  be a right angled triangle, having  $BAC$  a right angle : Book VI.  
 I say that the figure described upon  $BC$  is equal to the similar and similarly described figures upon  $BA$ ,  $AC$ .

Let the perpendicular  $AD$  be drawn.

Wherefore because, in the right angled triangle  $ABC$ ,  $AD$  hath been drawn from the right angle at  $A$  perpendicular to the base  $BC$ ; therefore (by 8. 6.) the triangles  $ABD$ ,  $ADC$ , at the perpendicular, are similar to the whole  $ABC$  and to one another :



and because  $ABC$  is similar to  $ABD$ ; therefore it is, as  $CB$  is to  $BA$  so is  $AB$  to  $BD$ : and because the three straight lines are proportionals; it is, (by 2. Cor. to 20. 6.) as the first is to the third so is the figure upon the first to a similar and similarly described figure upon the second; wherefore as  $CB$  is to  $BD$  so is the figure upon  $CB$  to the similar and similarly described figure upon  $AB$ . Certainly for the same reason also as  $BC$  is to  $CD$  so is the figure upon  $CB$  to the figure upon  $AC$ : so that it is, (by 24. 5.) as  $BC$  is to  $BD$ ,  $DC$  so is the figure upon  $BC$  to the figures similar and similarly described upon  $BA$ ,  $AC$ : but  $BC$  is equal to  $BD$ ,  $DC$ ; therefore the figure upon  $BC$  is equal to the figures similar and similarly described upon  $BA$ ,  $AC$ .

Wherefore in right angled triangles, the figure described upon the side subtending the right angle is equal to the figures similar and similarly described upon the sides containing the right angle. Which was to be demonstrated.

OTHERWISE. Because similar figures are (by 20. 6.) in the duplicate ratio of the sides of like ratio, therefore the figure upon  $BC$  has to the figure upon  $BA$  the duplicate ratio of that which  $BC$  has to  $BA$ ; but (by Cor. 1. to 20. 6.) the square of  $BC$  also has to the square of  $BA$  the duplicate ratio of that which  $BC$  has to  $BA$ ; therefore also (by 11. 5.) as the figure upon  $BC$  is to the figure upon  $BA$  so is the square of  $BC$  to the square of  $BA$ . Certainly for the same reason, also as the figure upon  $BC$  is to the figure upon  $CA$  so is the square of  $BC$  to the square of  $CA$ : so that also (by 24. 5.) as the figure upon  $BC$  is to the figures upon  $BA$ ,  $AC$  so is the square of  $BC$  to the squares of  $BA$ ,  $AC$ : but the square of  $BC$  is equal (by 47. 1.) to the squares of  $BA$ ,  $AC$ ; therefore also



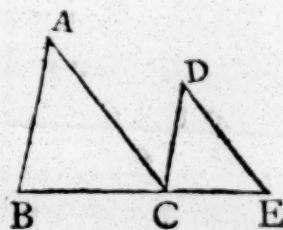
Book VI. the figure upon BC is equal to the figures similar and similarly described upon BA, AC. Which was to be demonstrated.

## P R O P. XXXII.

If two triangles, having two sides proportional to two sides be joined together at one angle, so as that their sides of like ratio may be also parallel; the remaining sides of the triangles will be in a straight line.

Let ABC, DCE be two triangles having the two sides BA, AC proportional to the two sides CD, DE: so that BA *may be* to AC as CD *is* to DE; and AB parallel to CD and AC to DE: I say that BC is in a straight line with CE.

For because AB is parallel to DC, and the straight line AC hath fallen upon them, the alternate angles (by 29. 1.) BAC, ACD are equal to one another: Certainly for the same reason the *angle* CDE is equal to ACD: so that also (by com. not. 1.) the *angle* BAC is equal to CDE: And because there are two triangles ABC, DCE having one angle the *angle* at A equal to one angle the *angle* at D and the sides (by supp.) about the equal angles proportional; as BA *is* to AC so *is* CD to DE; therefore (by 6. 6.) the triangle ABC is equiangular to the triangle DCE: therefore the angle ABC is equal to DCE; and also the *angle* ACD has been demonstrated *to be* equal to BAC: therefore the whole angle ACE is equal to the two angles ABC, BAC; let the common angle ACB be added; therefore the *two* angles ACE, ACB are equal to the *three* angles BAC, ACB, ABC; but (by 32. 1.) the *three* angles BAC, ACB, ABC are equal to two right angles; therefore also the angles ACE, ACB are equal to two right angles: with a certain straight line AC and at the point C in it, two straight lines BC, CE not being towards the same parts, make the adjacent angles ACE, ACB equal to two right angles; therefore (by 14. 1.) BC is in a straight line with CE.



Wherefore if two triangles, having two sides proportional to two sides be joined together at one angle; so as that their sides of like ratio may be parallel: the remaining sides of the triangles will be in a straight line. Which was to be demonstrated.

P R O P.

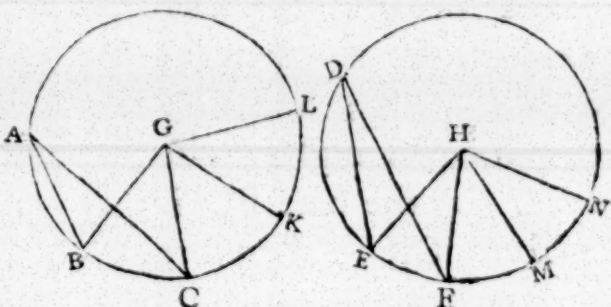


In equal circles, the angles have the same ratio *to one another* as the circumferences on which they stand, whether they stand at the center or at the circumference. And besides also the sectors *have the same proportion* because they stand at the centers.

Let ABC, DEF be equal circles; and let the angles BGC, EHF be at their centers G, H: and the angles BAC, EDF at the circumferences: I say that it is, as the circumference BC is to the circumference EF so is both the angle BGC to the angle EHF and the angle BAC to the angle EDF: and besides so is the sector GBC to the sector HEF.

Take any number of contiguous *circumferences* (by 1. 4.) CK, KL equal to the circumference BC, and also any number of *circumferences which may accidentally happen* FM, MN equal to the circumference EF; and let GK, GL; HM, HN be joined

Wherefore because the circumferences BC, CK, KL are equal to one another; also (by 27. 3.) the angles BGC, CGK, KGL are equal to one another: whatsoever multiple therefore the circumference BL is of the circumference BC; the same multiple also is the angle BGL of the angle BGC. Certainly for the same reason also, whatsoever multiple the circumference EN is of the circumference EF; the same multiple also is the angle EHN of the angle EHF: and if the circumference BL be equal to the circumference EN; also (by 27. 3.) the angle BGL is equal to the angle EHN: and if the circumference BL be greater than the circumference EN; also the angle BGL is greater than the angle EHN; and if less, less: there being four magnitudes; the two circumferences BC, EF; and the two angles BGC, EHF; there have been taken the circumference BL and the angle BGL equimultiples of the circumference BC and of the angle BGC; and the circumference EN and the angle EHN *any other equimultiples which may accidentally happen* of the circumference EF and of the angle EHF; and it has been demonstrated that if the circumference BL exceed





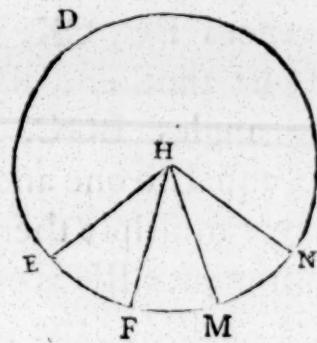
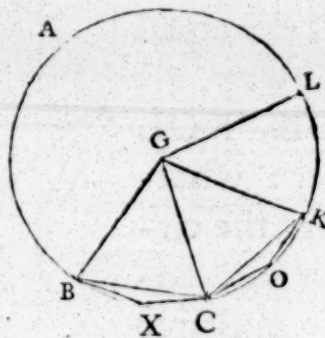
Book VI. exceed the circumference EN also the angle BGL exceeds the angle EHN : and if equal, equal : and if less, less : it is therefore (by 5. def. 5.) as the circumference BC is to the circumference EF so is the angle BGC to the angle EHF : but as the angle BGC is to the angle EHF so is (by 15. 5.) the angle BAC to the angle EDF ; for they are (by 20. 3.) the doubles, each of each : therefore as the circumference BC is to the circumference EF so is both the angle BGC to the angle EHF ; and the angle BAC to the angle EDF.

Wherefore in equal circles, the angles have the same ratio to one another as the circumferences on which they stand ; whether they stand at the centers or at the circumferences. Which was to be demonstrated.

I say also, that as the circumference BC is to the circumference EF so is the sector GBC to the sector HEF.

For let BC, CK be joined ; and the points X, O being taken in the circumferences BC, CK ; let BX, XC ; CO, OK be joined.

And because the two BG, GC are equal to the two CG, GK ; and they contain equal angles ; also (by 4. 1.) the base BC is equal to CK ; therefore also the triangle BGC is equal to the triangle GCK : and because the circumfe-



rence BC is equal to the circumference CK ; also the remaining circumference, which makes up the whole circle ABC, is equal to the remaining circumference which makes up the same circle ; so that also (by 27. 3.) the angle BXC is equal to the angle COK ; wherefore (by 11. def. 3.) the segment BXC is similar to the segment COK ; and they are upon equal straight lines BC, CK : but similar segments of circles upon equal straight lines (by 24. 3.) are equal to one another : therefore the segment BXC is equal to the segment COK : but the triangle BGC is also equal to the triangle CGK ; therefore the whole sector GBC is equal to the whole sector GCK. Certainly for the same reason also the sector GKL is equal to either GKC or GCB : therefore the three sectors GBC, GCK, GKL are equal to one another.

Certainly



Certainly for the same reason also, the sectors HEF, HFM, HMN are equal to one another; whatsoever multiple therefore the circumference BL is of the circumference BC; the sector GBL is also the same multiple of the sector GBC. Certainly for the same reason also, whatsoever multiple the circumference EN is of the circumference EF; the sector HEN is also the same multiple of the sector HEF: And if the circumference BL be equal to the circumference EN; also the sector GBL is equal to the sector HEN: and if the circumference BL exceed the circumference EN; the sector GBL also exceeds the sector HEN; and if *the one* is deficient, *the other* is deficient. There being then four magnitudes, the two circumferences BC, EF; and the two sectors GBC, HEF; and equimultiples have been taken of the circumference BC, and of the sector GBC, viz. the circumference BL and the sector GBL; and equimultiples also of the circumference EF, and of the sector HEF, viz. the circumference EN and the sector HEN: and it hath been demonstrated, that if the circumference BL exceed the circumference EN, the sector GBL also exceeds the sector HEN; and if equal, equal; and if *the one* is deficient *the other* is deficient. Therefore it is (by 5. def. 5.) as the circumference BC is to the circumference EF so is the sector GBC to the sector HEF.

Cor. And it is manifest (by 11. 5.) also, that as the sector is to the sector so is the angle to the angle.

THE END OF THE SIXTH BOOK.



## THE CONCLUSION.

AFTER the first six books of these elements are read, it will be very proper for readers of all denominations, to make some kind of estimate of their improvement in the knowledge of magnitude, by taking a review of the whole. And this will be done with some astonishment by the judicious reader, when he considers to what a height he has raised himself above the vulgar in his conceptions upon this subject; or indeed even above himself, when he compares the common perception which he set out with, that *Magnitudes which exactly agree together are equal*, with the power of making a rectilineal figure equal to *any* given rectilineal figure, and similar to *any* other which may be given. But as an admiration of the great abilities of the author, and of his judicious arrangement will be the necessary consequence of right notions upon this subject; I shall decline all encomiums upon these, to turn my thoughts to the assistance of those who may not have reason to be so well satisfied with their improvement; which is likely to be most effectually given, by an enumeration of those circumstances, to which their want of success is most probably owing; an attention to which, upon some future perusal, may remove every defect. And although I have mentioned them in the course of these dissertations yet it may be useful as a conclusion to the whole to leave such an impression upon the reader's mind, as a summary of the whole is likely to make. The reader therefore is to consider first of all whether he may not have had all this time, even a wrong notion of the very subject. The subject is magnitude, and not number; nor is it sufficient for him to be satisfied that *things* equal to the same are equal to one another; but he must convince himself, *not give his assent*, that magnitudes which are equal to the same, are equal to one another; and this is to be done by a particular examination, of as many different kinds of  
magnitudes



magnitudes as he can easily recollect, especially of such, as are thus compared in these first six books. Next let him examine whether he has a proper notion of a definition, which he must do his endeavour to collect both from its nature and use; still keeping in his mind that every thing here refers to magnitude. And the shewing that any property is consistent with the definition is called a demonstration. But several demonstrations ought to be carefully examined by a student in the circumstances of the person to whom I am now writing, entirely with a view to see whether he understands what is meant by a geometrical demonstration; and as we suppose him to have his choice of the first six books; he ought especially to pick out the indirect demonstrations, which are of all others the most elegant and forcible, because they convince you contrary to the representation of the figure: the evidence of such propositions therefore, if its force be perceived at all, comes to the mind purer, as it is not affected by the prejudices which the reasoning upon a particular figure, which seems to represent the consequences, is but too apt to introduce. For such reasonings are often disturbed even by turning the figure upsidedown; and I myself have often puzzled a student sufficiently acute, by such trials as joining AD in the twenty first proposition of the first book, and requiring him to prove that AD and DC together are less than AB and BC together &c. The indirect demonstrations therefore are of all others the properest for a student to exercise himself in; as this will be the means of removing that very great difficulty which learners find in making a ready and proper use of the supposition; as his chief dependence for conviction in the indirect demonstrations, must be upon the supposition. It is true this may be got as I have remarked before, by an examination of the supposition in every proposition, especially such as consist of several circumstances; and varying them, one by one, observing the effect which every variation has upon demonstration; for by such a proceeding the force and use of the different parts of the supposition must be discovered, and indeed hardly by any other means. And in this progress he will be very much assisted by describing his figures with all the variety of positions, which his instruments and the supposition will admit of, until he is certain that he has conquered



quered every prejudice which can arise from reasoning upon a particular figure.

The reader is also carefully to observe the different methods of comparison made use of in the first six books. In the first book, the magnitudes are compared either; as exactly agreeing together; or as being equal to the same magnitude; or as being radius's of the same circle; which are all common and familiar notions. But in the second book, the method of comparison is by the rectangle, which is by no means a common notion, either as to its name or nature; and therefore should be made familiar; as well as perfectly understood. In some part of the third book, the method of comparison is by similar segments, which is also an artificial idea, and ought for the same reason to be made familiar. But the most artificial and therefore the most difficult of all, is that method of comparison founded upon a consideration of quantities as parts and multiples. It will be particularly useful for the student, who has failed in his first attempt to understand this fifth book, after reading carefully what I have said in explanation of the two first definitions, to take care to have the suppositions given him by another; and then carefully to perform every construction himself. And indeed the reader who would take the easiest way of making himself master of this subject, should have the ruler and compasses in his hand, through the first six books, only taking care not to forget, that he is handling a ruler and compasses.

THE END OF THE FIRST VOLUME.



